

# REGULARITY ESTIMATES IN HÖLDER SPACES FOR SCHRÖDINGER OPERATORS VIA A $T1$ THEOREM

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**ABSTRACT.** We derive Hölder regularity estimates for operators associated with a time independent Schrödinger operator of the form  $-\Delta + V$ . The results are obtained by checking a certain condition to the function  $T1$ . Our general method applies to get regularity estimates for maximal operators and square functions of the heat and Poisson semigroups, for Laplace transform type multipliers and also for Riesz transforms and negative powers  $(-\Delta + V)^{-\gamma/2}$ , all of them in an unified way.

## 1. INTRODUCTION AND STATEMENT OF THE RESULTS

Regularity estimates for second order differential operators are central in the theory of PDEs. In this context, Schauder estimates are fundamental results. They can be seen as boundedness between Hölder spaces of negative powers of operators.

In this paper we study regularity estimates in the Hölder classes  $C_{\mathcal{L}}^{0,\alpha}$ ,  $0 < \alpha < 1$ , of operators associated with the time independent Schrödinger operator in  $\mathbb{R}^n$ ,  $n \geq 3$ ,

$$\mathcal{L} := -\Delta + V.$$

The nonnegative potential  $V$  satisfies a reverse Hölder inequality for some  $q \geq n/2$ , see Section 2.

It is well-known that the classical Hölder space  $C^\alpha(\mathbb{R}^n)$  can be identified with the Campanato space  $BMO^\alpha$ , see [7]. In the Schrödinger case the analogous result was proved by B. Bongioanni, E. Harboure and O. Salinas in [6]. They identified the Hölder space associated to  $\mathcal{L}$  with a Campanato type  $BMO_{\mathcal{L}}^\alpha$  space, see Proposition 2.4 below. Therefore, in order to study regularity estimates we can take advantage of this characterization. In fact we shall present our results as boundedness of operators between  $BMO_{\mathcal{L}}^\alpha$  spaces.

The main point of this paper is to give a simple  $T1$  criterion for boundedness in  $BMO_{\mathcal{L}}^\alpha$  of the so-called  $\gamma$ -Schrödinger-Calderón-Zygmund operators  $T$ , see Definition 3.1. The advantage of this criterion is that everything reduces to check a certain condition on the function  $T1$ . The method is applied to the maximal operators associated with the semigroups  $e^{-t\mathcal{L}}$  and  $e^{-t\mathcal{L}^{1/2}}$  (or more general Poisson operators associated to the extension problem for  $\mathcal{L}^\sigma$ ), the  $\mathcal{L}$ -square functions, the Laplace transform type multipliers  $m(\mathcal{L})$ , the  $\mathcal{L}$ -Riesz transforms and the negative powers  $\mathcal{L}^{-\gamma/2}$ ,  $\gamma > 0$ .

We use the notation  $f_B = \frac{1}{|B|} \int_B f$ . The first result reads as follows.

**Theorem 1.1** ( $T1$ -type criterion for  $BMO_{\mathcal{L}}^\alpha$ ,  $0 < \alpha < 1$ ). *Let  $T$  be a  $\gamma$ -Schrödinger-Calderón-Zygmund operator,  $\gamma \geq 0$ , with smoothness exponent  $\delta$ , such that  $\alpha + \gamma < \min\{1, \delta\}$ . Then  $T$  is bounded from  $BMO_{\mathcal{L}}^\alpha$  into  $BMO_{\mathcal{L}}^{\alpha+\gamma}$  if and only if there exists a constant  $C$  such that*

$$\left(\frac{\rho(x)}{s}\right)^\alpha \frac{1}{|B|^{1+\frac{\gamma}{n}}} \int_B |T1(y) - (T1)_B| dy \leq C,$$

for every ball  $B = B(x, s)$ ,  $x \in \mathbb{R}^n$  and  $0 < s \leq \frac{1}{2}\rho(x)$ . Here  $\rho(x)$  is defined in (2.1).

We can also consider the endpoint case  $\alpha = 0$ .

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**Theorem 1.2** (*T1-type criterion for  $BMO_{\mathcal{L}}$* ). *Let  $T$  be a  $\gamma$ -Schrödinger-Calderón-Zygmund operator,  $0 \leq \gamma < \min\{1, \delta\}$ , with smoothness exponent  $\delta$ . Then  $T$  is a bounded operator from  $BMO_{\mathcal{L}}$  into  $BMO_{\mathcal{L}}^\gamma$  if and only if there exists a constant  $C$  such that*

$$\log \left( \frac{\rho(x)}{s} \right) \frac{1}{|B|^{1+\frac{\gamma}{n}}} \int_B |T1(y) - (T1)_B| dy \leq C,$$

for every ball  $B = B(x, s)$ ,  $x \in \mathbb{R}^n$  and  $0 < s \leq \frac{1}{2}\rho(x)$ .

Observe that for any  $x \in \mathbb{R}^n$  and  $0 < \alpha \leq 1$ , if  $0 < s \leq \frac{1}{2}\rho(x)$  then  $1 + \log \frac{\rho(x)}{s} \sim \log \frac{\rho(x)}{s}$  and  $1 + \frac{2^\alpha((\frac{\rho(x)}{s})^\alpha - 1)}{2^\alpha - 1} \sim \left(\frac{\rho(x)}{s}\right)^\alpha$ . Therefore, tracking down the exact constants in the proof we can see that Theorem 1.2 is indeed the limit case of Theorem 1.1.

Theorem 1.2 is a generalization of the T1-type criterion given in [2] for the case of the harmonic oscillator  $H = -\Delta + |x|^2$ . Here we require the dimension to be  $n \geq 3$ , while in [2] the dimension can be any  $n \geq 1$ .

As a by-product of our main results we are able to characterize pointwise multipliers of the spaces  $BMO_{\mathcal{L}}^\alpha$ , see Proposition 3.2 below. For pointwise multipliers of the classical  $BMO^\alpha$  spaces see the papers by S. Bloom [3], S. Janson [16] and E. Nakai and K. Yabuta [19].

Next we present the announced applications. For the definitions of the operators we kindly ask the reader to see subsections 4.1 to 4.5.

**Theorem 1.3.** *Let  $0 \leq \alpha < \min\{1, 2 - \frac{n}{q}\}$ . The maximal operators associated with the heat semigroup  $\{\mathcal{W}_t\}_{t>0}$  and with the generalized Poisson operators  $\{\mathcal{P}_t^\sigma\}_{t>0}$ , the Littlewood-Paley  $g$ -functions given in terms of the heat and the Poisson semigroups, and the Laplace transform type multipliers  $m(\mathcal{L})$ , are bounded from  $BMO_{\mathcal{L}}^\alpha$  into itself.*

In [9] it was proved that the maximal operator of the heat semigroup, the maximal operator of the Poisson semigroup and the square function of the heat semigroup are bounded in  $BMO_{\mathcal{L}}$ , and that the fractional integral  $\mathcal{L}^{-\gamma/2}$  maps  $L^{n/\gamma}(\mathbb{R}^n)$  into  $BMO_{\mathcal{L}}$ ,  $0 < \gamma < n$ . The square function was also studied in [1]. In [23] it was proved that the fractional integral in the case of the harmonic oscillator has similar boundedness properties in the scale of spaces  $BMO_H^\alpha$ , or more generally,  $C_H^{k,\alpha}(\mathbb{R}^n)$ .

The Riesz transforms associated to  $\mathcal{L}$  were introduced and studied in  $L^p(\mathbb{R}^n)$  in the seminal paper by Z. Shen [20]. B. Bongioanni, E. Harboure and O. Salinas developed their mapping properties on  $BMO_{\mathcal{L}}^\alpha$  in [5]. They also studied the corresponding boundedness results for the negative powers, see [6], and  $L^p$ -boundedness for the commutators with a function, see [4]. Following the pattern of the proof of Theorem 1.3 we can recover the results from [6] and [5]. We state them as a theorem for further reference.

**Theorem 1.4.** *Let  $\alpha \geq 0$  and  $0 < \gamma < n$ . Then:*

- *The  $\mathcal{L}$ -Riesz transforms are bounded from  $BMO_{\mathcal{L}}^\alpha$  into itself, for any  $0 \leq \alpha < 1 - \frac{n}{q}$ , with  $q > n$ .*
- *The negative powers  $\mathcal{L}^{-\gamma/2}$  are bounded from  $BMO_{\mathcal{L}}^\alpha$  into  $BMO_{\mathcal{L}}^{\alpha+\gamma}$  for  $\alpha + \gamma < \min\{1, 2 - \frac{n}{q}\}$ .*

The use of the action of an operator  $T$  on the function 1 in order to get some boundedness properties of  $T$  goes back to the celebrated work by G. David and J.-L. Journé, see [8]. For vector-valued versions of these criteria, see the papers by T. Hytönen [14] and T. Hytönen and L. Weis [15].

The paper is organized as follows. In section 2 we collect the technical results about the space  $BMO_{\mathcal{L}}^\alpha$ . Section 3 is devoted to the proofs of the main theorems. The applications are given in section 4. Through the paper the letters  $C$  and  $c$  denote positive constants that may change at each occurrence and  $\mathcal{S}$  is the class of rapidly decreasing  $C^\infty$  functions in  $\mathbb{R}^n$ .

## 2. THE SPACES $BMO_{\mathcal{L}}^\alpha$ , $0 \leq \alpha \leq 1$

The nonnegative potential  $V$  satisfies a reverse Hölder inequality for some  $q \geq \frac{n}{2}$ ; that is, there exists a constant  $C = C(q, V)$  such that

$$\left( \frac{1}{|B|} \int_B V(y)^q dy \right)^{1/q} \leq \frac{C}{|B|} \int_B V(y) dy,$$

for all balls  $B \subset \mathbb{R}^n$ . We write  $V \in RH_q$ . Associated to this potential, Z. Shen defines the critical radii function in [20] as

$$(2.1) \quad \rho(x) := \sup \left\{ r > 0 : \frac{1}{r^{n-2}} \int_{B(x,r)} V(y) dy \leq 1 \right\}, \quad x \in \mathbb{R}^n.$$

We have  $0 < \rho(x) < \infty$ .

Let us begin with some properties of the critical radii function  $\rho$ .

**Lemma 2.1** (See [20, Lemma 1.4]). *There exist  $c > 0$  and  $k_0 \geq 1$  such that for all  $x, y \in \mathbb{R}^n$ ,*

$$c^{-1} \rho(x) \left( 1 + \frac{|x-y|}{\rho(x)} \right)^{-k_0} \leq \rho(y) \leq c \rho(x) \left( 1 + \frac{|x-y|}{\rho(x)} \right)^{\frac{k_0}{k_0+1}}.$$

*In particular, there exists a positive constant  $C_1 < 1$  such that if  $|x-y| \leq \rho(x)$  then  $C_1 \rho(x) < \rho(y) < C_1^{-1} \rho(x)$ .*

**Covering by critical balls.** According to [10, Lemma 2.3] there exists a sequence of points  $\{x_k\}_{k=1}^\infty$  in  $\mathbb{R}^n$  such that if  $Q_k := B(x_k, \rho(x_k))$ ,  $k \in \mathbb{N}$ , then

- (a)  $\cup_{k=1}^\infty Q_k = \mathbb{R}^n$ , and
- (b) there exists  $N \in \mathbb{N}$  such that  $\text{card}\{j \in \mathbb{N} : Q_j^{**} \cap Q_k^{**} \neq \emptyset\} \leq N$ , for every  $k \in \mathbb{N}$ .

For a ball  $B$ , the notation  $B^*$  above means the ball with the same center as  $B$  and twice radius.

The definition of space  $BMO_{\mathcal{L}}$  was given in [9]. The space  $BMO_{\mathcal{L}}^\alpha$ ,  $0 < \alpha \leq 1$ , was introduced in [6]. We collect from there the following facts.

A locally integrable function  $f$  in  $\mathbb{R}^n$  is in  $BMO_{\mathcal{L}}^\alpha$ ,  $0 \leq \alpha \leq 1$  provided there exists  $C > 0$  such that

- (i)  $\frac{1}{|B|} \int_B |f(x) - f_B| dx \leq C |B|^{\frac{\alpha}{n}}$ , for every ball  $B$  in  $\mathbb{R}^n$ , and
- (ii)  $\frac{1}{|B|} \int_B |f(x)| dx \leq C |B|^{\frac{\alpha}{n}}$ , for every  $B = B(x_0, r_0)$ , where  $x_0 \in \mathbb{R}^n$  and  $r_0 \geq \rho(x_0)$ .

The norm  $\|f\|_{BMO_{\mathcal{L}}^\alpha}$  of  $f$  is defined as the minimum  $C > 0$  such that (i) and (ii) above hold. We have  $BMO_{\mathcal{L}}^0 = BMO_{\mathcal{L}}$ .

By using the classical John-Nirenberg inequality it can be seen that if in (i) and (ii)  $L^1$ -norms are replaced by  $L^p$ -norms, for  $1 < p < \infty$ , then the space  $BMO_{\mathcal{L}}^\alpha$  does not change and equivalent norms appear. In this case the conditions read:

- (i)<sub>p</sub>  $\left( \frac{1}{|B|} \int_B |f(x) - f_B|^p dx \right)^{1/p} \leq C |B|^{\frac{\alpha}{n}}$ , for every ball  $B$  in  $\mathbb{R}^n$ , and
- (ii)<sub>p</sub>  $\left( \frac{1}{|B|} \int_B |f(x)|^p dx \right)^{1/p} \leq C |B|^{\frac{\alpha}{n}}$ , for every  $B = B(x_0, r_0)$ , where  $x_0 \in \mathbb{R}^n$  and  $r_0 \geq \rho(x_0)$ .

Let us note that if (ii) (resp. (ii)<sub>p</sub>) above is true for some ball  $B$  then (i) (resp. (i)<sub>p</sub>) holds for the same ball, so we might ask to (i) (resp. (i)<sub>p</sub>) only for balls with radii smaller than  $\rho(x)$ .

The restriction  $\alpha \leq 1$  in the definition above is necessary because if  $\alpha > 1$  then the space only contains constant functions.

**Proposition 2.2.** *Let  $B = B(x, r)$  with  $r < \rho(x)$ .*

- (1) (See [9, Lemma 2]) *If  $f \in BMO_{\mathcal{L}}$  then  $|f_B| \leq C \left( 1 + \log \frac{\rho(x)}{r} \right) \|f\|_{BMO_{\mathcal{L}}}$ .*
- (2) (See [18, Proposition 4.3]) *If  $f \in BMO_{\mathcal{L}}^\alpha$ ,  $0 < \alpha \leq 1$ , then  $|f_B| \leq C_\alpha \|f\|_{BMO_{\mathcal{L}}^\alpha} \rho(x)^\alpha$ .*
- (3) (See [6, Proposition 3]) *A function  $f$  belongs to  $BMO_{\mathcal{L}}^\alpha$ ,  $0 \leq \alpha \leq 1$ , if and only if  $f$  satisfies (i) for every ball  $B = B(x_0, r_0)$  with  $r_0 < \rho(x_0)$  and  $|f|_{Q_k} \leq C |Q_k|^{1+\frac{\alpha}{n}}$ , for all balls  $Q_k$  given in the covering by critical balls above.*

**Lemma 2.3** (Boundedness criterion). *Let  $S$  be a linear operator defined on  $BMO_{\mathcal{L}}^\alpha$ ,  $0 \leq \alpha \leq 1$ . Then  $S$  is bounded from  $BMO_{\mathcal{L}}^\alpha$  into  $BMO_{\mathcal{L}}^{\alpha+\gamma}$ ,  $\alpha + \gamma \leq 1$ ,  $\gamma \geq 0$ , if there exists  $C > 0$  such that, for every  $f \in BMO_{\mathcal{L}}^\alpha$  and  $k \in \mathbb{N}$ ,*

- (A<sub>k</sub>)  $\frac{1}{|Q_k|^{1+\frac{\alpha+\gamma}{n}}} \int_{Q_k} |Sf(x)| dx \leq C \|f\|_{BMO_{\mathcal{L}}^{\alpha}}$ , and  
 (B<sub>k</sub>)  $\|Sf\|_{BMO^{\alpha+\gamma}(Q_k^*)} \leq C \|f\|_{BMO_{\mathcal{L}}^{\alpha}}$ , where  $BMO^{\alpha}(Q_k^*)$  denotes the usual  $BMO^{\alpha}$  space on the ball  $Q_k^*$ .

*Proof.* For  $\alpha = 0$  the result is already contained in [9, p. 346]. The general statement follows immediately from the definition of  $BMO_{\mathcal{L}}^{\alpha}$  and Lemma 2.1 (see Proposition 2.2).  $\square$

The duality of the  $\mathcal{L}$ -Hardy space  $H_{\mathcal{L}}^1$  with  $BMO_{\mathcal{L}}$  was proved in [9]. As mentioned in [6], the  $BMO_{\mathcal{L}}^{\alpha}$  spaces are the duals of the  $H_{\mathcal{L}}^p$  spaces defined in [10, 11, 12]. In fact, if  $s > n$  and  $0 \leq \alpha < 1$  then the dual of  $H_{\mathcal{L}}^{\frac{n}{n+\alpha}}$  is  $BMO_{\mathcal{L}}^{\alpha}$ , see also [13].

We denote by  $C^{\alpha}(\mathbb{R}^n)$  the space of  $\alpha$ -Hölder continuous functions on  $\mathbb{R}^n$  and by  $[f]_{C^{\alpha}}$  its usual seminorm. Recall that  $BMO^{\alpha}(\mathbb{R}^n) = C^{\alpha}(\mathbb{R}^n)$  with  $\|f\|_{BMO^{\alpha}(\mathbb{R}^n)} \sim [f]_{C^{\alpha}}$ .

**Proposition 2.4** (Campanato-type description of  $BMO_{\mathcal{L}}^{\alpha}$ , see [6, Proposition 4]). *Let  $0 < \alpha \leq 1$ . A function  $f$  belongs to  $BMO_{\mathcal{L}}^{\alpha}$  if and only if  $f \in C^{\alpha}(\mathbb{R}^n)$  and  $|f(x)| \leq C\rho(x)^{\alpha}$ , for all  $x \in \mathbb{R}^n$ . Moreover,  $\|f\|_{BMO_{\mathcal{L}}^{\alpha}} \sim [f]_{C^{\alpha}(\mathbb{R}^n)} + \|f\rho^{-\alpha}\|_{L^{\infty}(\mathbb{R}^n)}$ .*

In the following lemma we present examples of families of functions indexed by  $x_0 \in \mathbb{R}^n$  and  $0 < s \leq \rho(x_0)$  that are uniformly bounded in  $BMO_{\mathcal{L}}^{\alpha}$ . They will be very useful in the sequel.

**Lemma 2.5.** *There exists constants  $C, C_{\alpha} > 0$  such that for every  $x_0 \in \mathbb{R}^n$  and  $0 < s \leq \rho(x_0)$ ,*

- (a) *the function  $g_{x_0,s}(x) := \chi_{[0,s]}(|x - x_0|) \log\left(\frac{\rho(x_0)}{s}\right) + \chi_{(s,\rho(x_0)]}(|x - x_0|) \log\left(\frac{\rho(x_0)}{|x - x_0|}\right)$ ,  $x \in \mathbb{R}^n$ , belongs to  $BMO_{\mathcal{L}}$  and  $\|g_{x_0,s}\|_{BMO_{\mathcal{L}}} \leq C$ ;*  
 (b) *the function  $f_{x_0,s}(x) = \chi_{[0,s]}(|x - x_0|) (\rho(x_0)^{\alpha} - s^{\alpha}) + \chi_{(s,\rho(x_0)]}(|x - x_0|) (\rho(x_0)^{\alpha} - |x - x_0|^{\alpha})$ ,  $x \in \mathbb{R}^n$ , belongs to  $BMO_{\mathcal{L}}^{\alpha}$ ,  $0 < \alpha \leq 1$ , and  $\|f_{x_0,s}\|_{BMO_{\mathcal{L}}^{\alpha}} \leq C_{\alpha}$ .*

*Proof.* The proof of part (a) follows the same lines as the proof of Lemma 2.1 in [2]. We omit the details.

Let us continue with (b). Recall that the function  $h(x) = (1 - |x|^{\alpha}) \chi_{[0,1]}(|x|)$  is in  $BMO^{\alpha}(\mathbb{R}^n)$ . Hence, for every  $R > 0$ , the function  $h_R(x) := R^{\alpha} h(x/R)$  is in  $BMO^{\alpha}(\mathbb{R}^n)$  and  $\|h_R\|_{BMO^{\alpha}(\mathbb{R}^n)} \leq C$ , where  $C > 0$  is independent of  $R$ . Moreover, for every  $R > 0$  and  $S \geq 1$ , the function  $h_{R,S}(x) = \min\{R^{\alpha}(1 - S^{-\alpha}), R^{\alpha} h(x/R)\}$  belongs to  $BMO^{\alpha}(\mathbb{R}^n)$  and  $\|h_{R,S}\|_{BMO^{\alpha}(\mathbb{R}^n)} \leq C$ , where  $C > 0$  does not depend on  $R$  and  $S$ . Then, since for every  $x_0 \in \mathbb{R}^n$  and  $0 < s \leq \rho(x_0)$ ,

$$f_{x_0,s}(x) = h_{\rho(x_0), \frac{\rho(x_0)}{s}}(x - x_0), \quad x \in \mathbb{R}^n,$$

we get  $f_{x_0,s} \in BMO^{\alpha}(\mathbb{R}^n) = C^{\alpha}(\mathbb{R}^n)$  and  $\|f_{x_0,s}\|_{BMO^{\alpha}(\mathbb{R}^n)} \leq C$ . This, the obvious inequality  $|f_{x_0,s}(x)| \leq C\rho(x)^{\alpha}$ , for all  $x$ , uniformly in  $x_0$  and  $s \leq \rho(x_0)$ , and Proposition 2.4 imply the conclusion.  $\square$

### 3. OPERATORS AND PROOFS OF THE MAIN RESULTS

**3.1. The operators related to  $\mathcal{L}$ .** We denote by  $L_c^p(\mathbb{R}^n)$  the set of functions  $f \in L^p(\mathbb{R}^n)$ ,  $1 \leq p \leq \infty$ , whose support  $\text{supp}(f)$  is a compact subset of  $\mathbb{R}^n$ .

**Definition 3.1.** Let  $0 \leq \gamma < n$ ,  $1 < p \leq q < \infty$ ,  $\frac{1}{q} = \frac{1}{p} - \frac{\gamma}{n}$ . Let  $T$  be a bounded linear operator from  $L^p(\mathbb{R}^n)$  into  $L^q(\mathbb{R}^n)$  such that

$$Tf(x) = \int_{\mathbb{R}^n} K(x, y) f(y) dy, \quad f \in L_c^p(\mathbb{R}^n) \text{ and a.e. } x \notin \text{supp}(f).$$

We shall say that  $T$  is a  $\gamma$ -Schrödinger-Calderón-Zygmund operator with regularity exponent  $\delta > 0$  if for some constant  $C$

$$(1) \quad |K(x, y)| \leq \frac{C}{|x - y|^{n-\gamma}} \left(1 + \frac{|x - y|}{\rho(x)}\right)^{-N}, \text{ for all } N > 0 \text{ and } x \neq y,$$

$$(2) \quad |K(x, y) - K(x, z)| + |K(y, x) - K(z, x)| \leq C \frac{|y - z|^\delta}{|x - y|^{n-\gamma+\delta}}, \text{ when } |x - y| > 2|y - z|.$$

**Definition of  $Tf$  for  $f \in BMO_{\mathcal{L}}^\alpha$ ,  $0 \leq \alpha \leq 1$ .** Suppose that  $f \in BMO_{\mathcal{L}}^\alpha$  and  $R \geq \rho(x_0)$ ,  $x_0 \in \mathbb{R}^n$ . We define

$$Tf(x) = T(f\chi_{B(x_0, R)})(x) + \int_{B(x_0, R)^c} K(x, y)f(y) dy, \quad \text{a.e. } x \in B(x_0, R).$$

Note that first term in the right hand side makes sense since  $f\chi_{B(x_0, R)} \in L_c^p(\mathbb{R}^n)$ . The integral in the second term is absolutely convergent. Indeed, by Lemma 2.1, there exists a constant  $C$  such that for any  $x \in B(x_0, R)$ ,

$$\begin{aligned} \rho(x) &\leq c\rho(x_0) \left(1 + \frac{|x - x_0|}{\rho(x_0)}\right)^{\frac{k_0}{k_0+1}} \leq C \left(\rho(x_0) + \rho(x_0)^{1-\frac{k_0}{k_0+1}} |x - x_0|^{\frac{k_0}{k_0+1}}\right) \\ &\leq C \left(R + R^{1-\frac{k_0}{k_0+1}} |x - x_0|^{\frac{k_0}{k_0+1}}\right) \leq C2R. \end{aligned}$$

Hence, using the  $\gamma$ -Schrödinger-Calderón-Zygmund condition (1) for  $K$  with  $N + \gamma > \alpha$ ,

$$\begin{aligned} \int_{B(x_0, 2R)^c} |K(x, y)| |f(y)| dy &\leq C \sum_{j=1}^{\infty} \int_{2^j R < |y - x_0| \leq 2^{j+1} R} \frac{\rho(x)^N}{|x - y|^{n+N}} |f(y)| dy \\ (3.1) \quad &\leq C \sum_{j=1}^{\infty} \frac{\rho(x)^N}{(2^j R - R)^{n+N}} \int_{|y - x_0| \leq 2^{j+1} R} |f(y)| dy \\ &\leq CR^\alpha \|f\|_{BMO_{\mathcal{L}}^\alpha}, \quad \text{a.e. } x \in B(x_0, R). \end{aligned}$$

The definition of  $Tf(x)$  is also independent of  $R$  in the sense that if  $B(x_0, R) \subset B(x'_0, R')$ , with  $R' \geq \rho(x_0)$ , then the definition using  $B(x'_0, R')$  coincides almost everywhere in  $B(x_0, R)$  with the one just given, because in that situation,

$$\begin{aligned} T(f\chi_{B(x'_0, R')})(x) - T(f\chi_{B(x_0, R)})(x) &= T(f\chi_{B(x'_0, R') \setminus B(x_0, R)})(x) = \int_{B(x'_0, R') \setminus B(x_0, R)} K(x, y)f(y) dy \\ &= \int_{B(x_0, R)^c} K(x, y)f(y) dy - \int_{B(x'_0, R')^c} K(x, y)f(y) dy, \quad \text{a.e. } x \in B(x_0, R). \end{aligned}$$

The definition just given above is equally valid for  $f \equiv 1 \in BMO_{\mathcal{L}}$ .

Next we derive an expression for  $Tf$  where T1 appears that will be useful in the proof of our main results. Let  $x_0 \in \mathbb{R}^n$  and  $r_0 > 0$ . For  $B = B(x_0, r_0)$  we clearly have

$$(3.2) \quad f = (f - f_B)\chi_{B^{***}} + (f - f_B)\chi_{(B^{***})^c} + f_B =: f_1 + f_2 + f_3.$$

Let us choose  $R \geq \rho(x_0)$  such that  $B^{***} \subset B(x_0, R)$ . Using (3.2) we get

$$\begin{aligned} Tf(x) &= T(f\chi_{B(x_0, R)})(x) + \int_{B(x_0, R)^c} K(x, y)f(y) dy \\ &= T((f - f_B)\chi_{B^{***}})(x) + T((f - f_B)\chi_{B(x_0, R) \setminus B^{***}})(x) + f_B T(\chi_{B(x_0, R)})(x) \\ &\quad + \int_{B(x_0, R)^c} K(x, y)(f(y) - f_B) dy + f_B \int_{B(x_0, R)^c} K(x, y) dy \\ (3.3) \quad &= T((f - f_B)\chi_{B^{***}})(x) + \int_{(B^{***})^c} K(x, y)(f(y) - f_B) dy + f_B T1(x), \quad \text{a.e. } x \in B^{***}. \end{aligned}$$

We observe that there exists a constant  $C$  such that

$$(3.4) \quad \frac{1}{|B|^{1+\frac{\gamma}{n}}} \int_B |T1(y)| dy \leq C, \quad \text{for all } B = B(x, \rho(x)), \quad x \in \mathbb{R}^n.$$

Indeed, by Hölder's inequality and the  $L^p - L^q$  boundedness of  $T$ ,

$$\frac{1}{|B|^{1+\frac{\gamma}{n}}} \int_B |T(\chi_{B^*})(y)| dy \leq \frac{1}{|B|^{\frac{1}{q}+\frac{\gamma}{n}}} \left( \int_B |T(\chi_{B^*})(y)|^q dy \right)^{1/q} \leq C \frac{|B|^{1/p}}{|B|^{\frac{1}{q}+\frac{\gamma}{n}}} = C.$$

By the integral representation of  $T$  and the size condition (1) on  $K$  with  $N = n + \gamma$ , for  $y \in B(x, \rho(x))$  we have

$$\begin{aligned} |T(\chi_{(B^*)^c})(y)| &\leq C \sum_{k=1}^{\infty} \int_{2^j \rho(x) \leq |x-z| < 2^{j+1} \rho(x)} \frac{\rho(y)^{n+\gamma}}{|y-z|^{2n}} dz \\ &\leq C \rho(y)^{n+\gamma} \sum_{k=1}^{\infty} \frac{(2^{j+1} \rho(x))^n}{(2^j \rho(x) - \rho(x))^{2n}} \leq C \rho(x)^\gamma, \end{aligned}$$

because  $\rho(x) \sim \rho(y)$ . Thus (3.4) follows by linearity.

### 3.2. Proofs of Theorems 1.1 and 1.2.

*Proof of Theorem 1.1.* First we shall see that the condition on  $T1$  implies that  $T$  is bounded from  $BMO_{\mathcal{L}}^\alpha$  into  $BMO_{\mathcal{L}}^{\alpha+\gamma}$ . In order to do this, we will show that there exists  $C > 0$  such that the properties  $(A_k)$  and  $(B_k)$  stated in Lemma 2.3 hold for every  $k \in \mathbb{N}$  and  $f \in BMO_{\mathcal{L}}^\alpha$ .

We begin with  $(A_k)$ . According to (3.3) with  $B = Q_k$ ,

$$Tf(x) = T((f - f_{Q_k})\chi_{Q_k^{***}})(x) + \int_{(Q_k^{***})^c} K(x, y)(f(y) - f_{Q_k}) dy + f_{Q_k} T1(x), \quad \text{a.e. } x \in Q_k.$$

As  $T$  maps  $L^p(\mathbb{R}^n)$  into  $L^q(\mathbb{R}^n)$ ,  $\frac{1}{q} = \frac{1}{p} - \frac{\gamma}{n}$ , by Hölder's inequality,

$$\begin{aligned} \frac{1}{|Q_k|^{1+\frac{\alpha+\gamma}{n}}} \int_{Q_k} |T((f - f_{Q_k})\chi_{Q_k^{***}})(x)| dx &\leq \frac{1}{|Q_k|^{\frac{1}{q}+\frac{\alpha+\gamma}{n}}} \left( \int_{Q_k} |T((f - f_{Q_k})\chi_{Q_k^{***}})(x)|^q dx \right)^{1/q} \\ &\leq \frac{C}{|Q_k|^{\frac{\alpha}{n}}} \left( \frac{1}{|Q_k|} \int_{Q_k^{***}} |f(x) - f_{Q_k}|^p dx \right)^{1/p} \leq C \|f\|_{BMO_{\mathcal{L}}^\alpha}. \end{aligned}$$

On the other hand, given  $x \in Q_k$ , we have  $\rho(x) \sim \rho(x_k)$  and if  $|x_k - y| > 2^j \rho(x_k)$ ,  $j \in \mathbb{N}$ , then  $|x - y| \geq 2^{j-1} \rho(x_k)$ . By the size condition (1) of the kernel  $K$ , for any  $N > \alpha$  we have

$$\begin{aligned} \frac{1}{|Q_k|^{\frac{\alpha+\gamma}{n}}} \left| \int_{(Q_k^{***})^c} K(x, y)(f(y) - f_{Q_k}) dy \right| &\leq \frac{1}{|Q_k|^{\frac{\alpha+\gamma}{n}}} \int_{(Q_k^{***})^c} |K(x, y)| |f(y) - f_{Q_k}| dy \\ &\leq \frac{C}{|Q_k|^{\frac{\alpha+\gamma}{n}}} \int_{(Q_k^{***})^c} \frac{1}{|x - y|^{n-\gamma}} \left( 1 + \frac{|x - y|}{\rho(x)} \right)^{-N} |f(y) - f_{Q_k}| dy \\ &\leq \frac{C}{|Q_k|^{\frac{\alpha+\gamma}{n}}} \sum_{j=3}^{\infty} \int_{2^j \rho(x_k) < |x_k - y| \leq 2^{j+1} \rho(x_k)} \frac{\rho(x)^N}{|x - y|^{n-\gamma+N}} |f(y) - f_{Q_k}| dy \\ &\leq \frac{C}{\rho(x_k)^\alpha} \sum_{j=3}^{\infty} \frac{\rho(x_k)^N}{(2^j \rho(x_k))^{n+N}} \int_{|x_k - y| \leq 2^{j+1} \rho(x_k)} |f(y) - f_{Q_k}| dy \\ &\leq C \sum_{j=3}^{\infty} 2^{-j(N-\alpha)} (j+1) \|f\|_{BMO_{\mathcal{L}}^\alpha} \leq C \|f\|_{BMO_{\mathcal{L}}^\alpha}. \end{aligned}$$

Finally, by (3.4),

$$\frac{1}{|Q_k|^{1+\frac{\alpha+\gamma}{n}}} \int_{Q_k} |f_{Q_k} T1(x)| dx = \frac{|f_{Q_k}|}{|Q_k|^{\frac{\alpha}{n}}} \frac{1}{|Q_k|^{1+\frac{\gamma}{n}}} \int_{Q_k} |T1(x)| dx \leq C \|f\|_{BMO_{\mathcal{L}}^\alpha}.$$

Hence, we conclude that  $(A_k)$  holds for  $T$  with a constant  $C$  that does not depend on  $k$ .

Let us continue with  $(B_k)$ . Let  $B = B(x_0, r_0) \subseteq Q_k^*$ , where  $x_0 \in \mathbb{R}^n$  and  $r_0 > 0$ . Note that if  $r_0 \geq \frac{1}{2}\rho(x_0)$  then  $\rho(x_0) \sim \rho(x_k) \sim r_0$ , so proceeding as above we have

$$\frac{1}{|B|^{1+\frac{\alpha+\gamma}{n}}} \int_B |Tf(x) - (Tf)_B| \, dx \leq \frac{2}{|B|^{1+\frac{\alpha+\gamma}{n}}} \int_B |Tf(x)| \, dx \leq C \|f\|_{BMO_{\mathcal{L}}^{\alpha}}.$$

Assume next that  $0 < r_0 < \frac{1}{2}\rho(x_0)$ . Using (3.3) we have that

$$\begin{aligned} \frac{1}{|B|^{1+\frac{\alpha+\gamma}{n}}} \int_B |Tf(x) - (Tf)_B| \, dx &\leq \frac{1}{|B|^{1+\frac{\alpha+\gamma}{n}}} \int_B \frac{1}{|B|} \int_B |Tf_1(x) - Tf_1(z)| \, dz \, dx \\ &\quad + \frac{1}{|B|^{1+\frac{\alpha+\gamma}{n}}} \int_B \frac{1}{|B|} \int_B |F_2(x) - F_2(z)| \, dz \, dx \\ &\quad + \frac{1}{|B|^{1+\frac{\alpha+\gamma}{n}}} \int_B |Tf_3(x) - (Tf_3)_B| \, dx =: L_1 + L_2 + L_3, \end{aligned}$$

where  $f = f_1 + f_2 + f_3$  as in (3.3) and we defined

$$F_2(x) = \int_{(B^{***})^c} K(x, y) f_2(y) \, dy, \quad x \in B.$$

Again Hölder's inequality and  $L^p - L^q$  boundedness of  $T$  give  $L_1 \leq C \|f\|_{BMO_{\mathcal{L}}^{\alpha}}$ . Let us estimate  $L_2$ . Take  $x, z \in B$  and  $y \in (B^{***})^c$ . Then  $8r_0 < |y - x_0| \leq |y - x| + r_0$  and therefore  $2|x - x_0| < 4r_0 < |y - x|$ . Under these conditions we can apply the smoothness of the kernel (2) and the restriction  $\alpha + \gamma < \min\{1, \delta\}$  to get

$$\begin{aligned} \frac{1}{|B|^{\frac{\alpha+\gamma}{n}}} |F_2(x) - F_2(z)| &\leq \frac{C}{r_0^{\alpha+\gamma}} \int_{(B^{***})^c} |K(x, y) - K(z, y)| |f(y) - f_B| \, dy \\ &\leq \frac{C}{r_0^{\alpha+\gamma}} \sum_{j=3}^{\infty} \int_{2^j r_0 \leq |x_0 - y| < 2^{j+1} r_0} \frac{|x - z|^{\delta}}{|x - y|^{n-\gamma+\delta}} |f(y) - f_B| \, dy \\ &\leq \frac{C}{r_0^{\alpha+\gamma}} \sum_{j=3}^{\infty} \frac{r_0^{\delta}}{((2^j - 1)r_0)^{n-\gamma+\delta}} \int_{2^j r_0 \leq |x_0 - y| < 2^{j+1} r_0} |f(y) - f_B| \, dy \\ &\leq C \sum_{j=3}^{\infty} \frac{2^{-j(\delta-(\alpha+\gamma))}}{(2^{j+1} r_0)^{n+\alpha}} \int_{|x_0 - y| < 2^{j+1} r_0} |f(y) - f_B| \, dy \\ &\leq C \|f\|_{BMO_{\mathcal{L}}^{\alpha}} \sum_{j=3}^{\infty} 2^{-j(\delta-(\alpha+\gamma))} (j+1) = C \|f\|_{BMO_{\mathcal{L}}^{\alpha}}. \end{aligned}$$

Therefore,  $L_2 \leq C \|f\|_{BMO_{\mathcal{L}}^{\alpha}}$ . We finally consider  $L_3$ . Using Proposition 2.2(2) and the assumption on  $T1$  it follows that

$$\begin{aligned} (3.5) \quad L_3 &= \frac{|f_B|}{|B|^{1+\frac{\alpha+\gamma}{n}}} \int_B |T1(x) - (T1)_B| \, dx \\ &\leq C \|f\|_{BMO_{\mathcal{L}}^{\alpha}} \left( \frac{\rho(x_0)}{r_0} \right)^{\alpha} \frac{1}{|B|^{1+\frac{\alpha}{n}}} \int_B |T1(x) - (T1)_B| \, dx \leq C \|f\|_{BMO_{\mathcal{L}}^{\alpha}}. \end{aligned}$$

This concludes the proof of  $(B_k)$ . Hence  $T$  is bounded from  $BMO_{\mathcal{L}}^{\alpha}$  into  $BMO_{\mathcal{L}}^{\alpha+\gamma}$ .

Let us now prove the converse statement. Suppose that  $T$  is bounded from  $BMO_{\mathcal{L}}^{\alpha}$  into  $BMO_{\mathcal{L}}^{\alpha+\gamma}$ . Let  $x_0 \in \mathbb{R}^n$  and  $0 < s \leq \frac{1}{2}\rho(x_0)$  and  $B = B(x_0, s)$ . For such  $x_0$  and  $s$  consider the nonnegative function  $f_0(x) \equiv f_{x_0, s}(x)$  defined in Lemma 2.5. Using the decomposition  $f_0 = (f_0 - (f_0)_B)\chi_{B^{***}} + (f_0 - (f_0)_B)\chi_{(B^{***})^c} + (f_0)_B =: f_1 + f_2 + (f_0)_B$  we can write  $(f_0)_B T1(y) = T f_0(y) - T f_1(y) - T f_2(y)$ , so

$$(f_0)_B \frac{1}{|B|^{1+\frac{\alpha+\gamma}{n}}} \int_B |T1(y) - T1_B| \, dy \leq \sum_{i=0}^2 \frac{1}{|B|^{1+\frac{\alpha+\gamma}{n}}} \int_B |T f_i(y) - (T f_i)_B| \, dy.$$



We can check that each of the three terms above is controlled by  $C \|f_0\|_{BMO_{\mathcal{L}}^{\alpha}} \leq C$ , where  $C$  is independent of  $x_0$  and  $s$ . Indeed, the case  $i = 0$  follows by the hypothesis about the boundedness of  $T$ . For  $i = 1$  the estimate follows, as usual, by Hölder's inequality and  $L^p - L^q$  boundedness of  $T$ . The term for  $i = 2$  is done as  $L_2$  above. Thus, since  $(f_0)_B = C(\rho(x_0)^{\alpha} - s^{\alpha})$  we obtain

$$\left(\frac{\rho(x_0)}{s}\right)^{\alpha} \frac{1}{|B|^{1+\frac{\alpha}{n}}} \int_B |T1(y) - (T1)_B| dy \leq C.$$

□

*Proof of Theorem 1.2.* The proof is the same as the proof of Theorem 1.1 putting  $\alpha = 0$  everywhere, except for just two differences. The first one is the estimate of the term  $L_3$ , where we must apply Proposition 2.2(1) instead of (2). The second difference appears is the proof of the converse, where instead of  $f_{x_0,s}(x)$  we have to consider the function  $g_{x_0,s}(x)$  of Lemma 2.5. □

### 3.3. Pointwise multipliers in $BMO_{\mathcal{L}}^{\alpha}$ , $0 \leq \alpha < 1$ .

**Proposition 3.2.** *Let  $\psi$  be a measurable function on  $\mathbb{R}^n$ . We denote by  $T_{\psi}$  the multiplier operator defined by  $T_{\psi}(f) = f\psi$ . Then*

- (A)  *$T_{\psi}$  is a bounded operator in  $BMO_{\mathcal{L}}$  if and only if  $\psi \in L^{\infty}(\mathbb{R}^n)$  and there exists  $C > 0$  such that, for all balls  $B = B(x_0, s)$  with  $0 < s < \frac{1}{2}\rho(x_0)$ ,*

$$\log\left(\frac{\rho(x_0)}{s}\right) \frac{1}{|B|} \int_B |\psi(y) - \psi_B| dy \leq C.$$

- (B)  *$T_{\psi}$  is a bounded operator in  $BMO_{\mathcal{L}}^{\alpha}$ ,  $0 < \alpha < 1$ , if and only if  $\psi \in L^{\infty}(\mathbb{R}^n)$  and there exists  $C > 0$  such that, for all balls  $B = B(x_0, s)$  with  $0 < s < \frac{1}{2}\rho(x_0)$ ,*

$$\left(\frac{\rho(x_0)}{s}\right)^{\alpha} \frac{1}{|B|} \int_B |\psi(y) - \psi_B| dy \leq C.$$

**Remark 3.3.** If  $\psi \in C^{0,\beta}(\mathbb{R}^n) \cap L^{\infty}(\mathbb{R}^n)$ ,  $0 < \beta \leq 1$ , then  $T_{\psi}$  is bounded on  $BMO_{\mathcal{L}}$ . Moreover, if for some  $\gamma$ -Schrödinger-Calderón-Zygmund operator  $T$  we have that  $T1$  defines a pointwise multiplier in  $BMO_{\mathcal{L}}^{\alpha}$  then the proposition above and Theorems 1.2 and 1.1 imply that  $T$  is a bounded operator on  $BMO_{\mathcal{L}}^{\alpha}$ .

*Proof of Proposition 3.2.* Let us first prove (B). Suppose that  $T_{\psi}$  is a bounded operator on  $BMO_{\mathcal{L}}^{\alpha}$ ,  $0 < \alpha < 1$ . For the function  $f_{x_0,s}(x)$  defined in Lemma 2.5 and any ball  $B = B(x_0, s)$  with  $0 < s \leq \frac{1}{2}\rho(x_0)$ , by Proposition 2.2(2) applied to  $f\psi$  and the hypothesis, we get

$$\begin{aligned} \left(\frac{\rho(x_0)}{s}\right)^{\alpha} \frac{1}{|B|} \int_B |\psi(x)| dx &\leq C_{\alpha} \frac{(\rho(x_0)^{\alpha} - s^{\alpha})}{|B|^{1+\frac{\alpha}{n}}} \int_B |\psi(x)| dx = \frac{C_{\alpha}}{|B|^{1+\frac{\alpha}{n}}} \int_B |\psi(x)f_{x_0,s}(x)| dx \\ &\leq \frac{C_{\alpha}}{|B|^{1+\frac{\alpha}{n}}} \int_B |(\psi f_{x_0,s})(x) - (\psi f_{x_0,s})_B| dx + \frac{C_{\alpha}}{|B|^{\frac{\alpha}{n}}} (\psi f_{x_0,s})_B \\ &\leq C_{\alpha} \|f_{x_0,s}\|_{BMO_{\mathcal{L}}^{\alpha}} + C_{\alpha} \left(\frac{\rho(x_0)}{s}\right)^{\alpha} \|\psi f_{x_0,s}\|_{BMO_{\mathcal{L}}^{\alpha}} \\ &\leq C_{\alpha} \left(\frac{\rho(x_0)}{s}\right)^{\alpha} \|f_{x_0,s}\|_{BMO_{\mathcal{L}}^{\alpha}} \leq C \left(\frac{\rho(x_0)}{s}\right)^{\alpha}. \end{aligned}$$

Hence  $|\psi|_B \leq C$  with  $C$  independent of  $B$ , so that  $\psi$  is bounded. Next we check the condition on  $\psi$ . We have

$$\begin{aligned} \left(\frac{\rho(x_0)}{s}\right)^{\alpha} \frac{1}{|B|} \int_B |\psi(x) - \psi_B| dx &\leq C_{\alpha} \frac{(\rho(x_0)^{\alpha} - s^{\alpha})}{|B|^{1+\frac{\alpha}{n}}} \int_B |\psi(x) - \psi_B| dx \\ &\leq \frac{C_{\alpha}}{|B|^{1+\frac{\alpha}{n}}} \int_B |\psi(x)f_{x_0,s}(x) - (\psi f_{x_0,s})_B| dx \\ &\leq C_{\alpha} \|\psi f_{x_0,s}\|_{BMO_{\mathcal{L}}^{\alpha}} \leq C_{\alpha} \|f_{x_0,s}\|_{BMO_{\mathcal{L}}^{\alpha}} \leq C. \end{aligned}$$



The constants  $C$  and  $C_\alpha$  appearing in this proof do not depend on  $x_0 \in \mathbb{R}^n$  and  $0 < s \leq \frac{1}{2}\rho(x_0)$ .

For the converse statement, assume  $\psi$  satisfies the properties required in the hypothesis. The kernel of the operator  $T = T_\psi$  is zero and  $T_\psi 1(x) = \psi(x)$ , so the conclusion follows by Theorem 1.1.

The proof of (A) is completely analogous by using the function  $g_{x_0,s}(x)$  of Lemma 2.5 instead of  $f_{x_0,s}(x)$  and by applying Theorem 1.2.  $\square$

#### 4. APPLICATIONS

In the following subsections, we prove Theorems 1.3 and 1.4. In order to adapt our results to the applications we need the following remark.

**Remark 4.1** (Vector-valued setting). Theorems 1.2 and 1.1 can also be stated in a vector valued setting. If  $Tf$  takes values in a Banach space  $\mathbb{B}$  and the absolute values in the conditions are replaced by the norm in  $\mathbb{B}$  then both results hold.

**4.1. Maximal operators for the heat–diffusion semigroup  $e^{-t\mathcal{L}}$ .** Let  $\{\mathcal{W}_t\}_{t>0}$  be the heat–diffusion semigroup associated to  $\mathcal{L}$ :

$$\mathcal{W}_t f(x) \equiv e^{-t\mathcal{L}} f(x) = \int_{\mathbb{R}^n} \mathcal{W}_t(x, y) f(y) dy, \quad f \in L^2(\mathbb{R}^n), \quad x \in \mathbb{R}^n, \quad t > 0.$$

The kernel of the classical heat semigroup  $\{W_t\}_{t>0} = \{e^{t\Delta}\}_{t>0}$  on  $\mathbb{R}^n$  is

$$W_t(x) := \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x|^2}{4t}}, \quad x \in \mathbb{R}^n, \quad t > 0,$$

In the following arguments we need some well known estimates about the kernel  $\mathcal{W}_t(x, y)$ .

**Lemma 4.2** (See [12, 17]). *For every  $N > 0$  there exists a constant  $C_N$  such that*

$$0 \leq \mathcal{W}_t(x, y) \leq C_N t^{-n/2} e^{-\frac{|x-y|^2}{5t}} \left(1 + \frac{\sqrt{t}}{\rho(x)} + \frac{\sqrt{t}}{\rho(y)}\right)^{-N}, \quad x, y \in \mathbb{R}^n, \quad t > 0.$$

**Lemma 4.3** (See [12, Proposition 2.16]). *There exists a nonnegative function  $\omega \in \mathcal{S}$  such that*

$$|\mathcal{W}_t(x, y) - \mathcal{W}_t(x - y)| \leq \left(\frac{\sqrt{t}}{\rho(x)}\right)^{\delta_0} \omega_t(x - y), \quad x, y \in \mathbb{R}^n, \quad t > 0,$$

where  $\omega_t(x - y) := t^{-n/2} \omega((x - y)/\sqrt{t})$  and

$$\delta_0 := 2 - \frac{n}{q} > 0.$$

In fact, going through the proof of [12] we see that  $\omega(x) = e^{-|x|^2}$ .

**Lemma 4.4** (See [11, Proposition 4.11]). *For every  $0 < \delta < \delta_0$ , there exists a constant  $c > 0$  such that for every  $N > 0$  there exists a constant  $C > 0$  such that for  $|y - z| < \sqrt{t}$  we have*

$$|\mathcal{W}_t(x, y) - \mathcal{W}_t(x, z)| \leq C \left(\frac{|y - z|}{\sqrt{t}}\right)^\delta t^{-n/2} e^{-c|x-y|^2/t} \left(1 + \frac{\sqrt{t}}{\rho(x)} + \frac{\sqrt{t}}{\rho(y)}\right)^{-N}.$$

**Lemma 4.5** (See [12, Proposition 2.17]). *For every  $0 < \delta < \min\{1, \delta_0\}$ ,*

$$|(\mathcal{W}_t(x, y) - \mathcal{W}_t(x - y)) - (\mathcal{W}_t(x, z) - \mathcal{W}_t(x - z))| \leq C \left(\frac{|y - z|}{\rho(x)}\right)^\delta \omega_t(x - y),$$

for all  $x, y \in \mathbb{R}^n$  and  $t > 0$ , with  $|y - z| < C\rho(y)$  and  $|y - z| < \frac{1}{4}|x - y|$ .

To prove that the maximal operator  $\mathcal{W}^*$  defined by  $\mathcal{W}^* f(x) = \sup_{t>0} |\mathcal{W}_t f(x)|$  is bounded from  $BMO_{\mathcal{L}}^\alpha$  into itself we give a vector-valued interpretation of the operator and apply Remark 4.1. Indeed, it is clear that  $\mathcal{W}^* f = \|\mathcal{W}_t f\|_E$ , with  $E = L^\infty((0, \infty), dt)$ . Hence, it is enough to show that the operator  $V(f) := (\mathcal{W}_t f)_{t>0}$  is bounded from  $BMO_{\mathcal{L}, E}^\alpha$  into itself, where the space  $BMO_{\mathcal{L}, E}^\alpha$  is defined in the obvious way by replacing the absolute values  $|\cdot|$  by norms  $\|\cdot\|_E$ .

By the Spectral Theorem,  $V$  is bounded from  $L^2(\mathbb{R}^n)$  into  $L_E^2(\mathbb{R}^n)$ . The desired result is then deduced from the following proposition.

**Proposition 4.6.** *Let  $x, y, z \in \mathbb{R}^n$  and  $N > 0$ . Then*

- (i)  $\|\mathcal{W}_t(x, y)\|_E \leq \frac{C}{|x-y|^n} \left(1 + \frac{|x-y|}{\rho(x)} + \frac{|x-y|}{\rho(y)}\right)^{-N};$
- (ii)  $\|\mathcal{W}_t(x, y) - \mathcal{W}_t(x, z)\|_E + \|\mathcal{W}_t(y, x) - \mathcal{W}_t(z, x)\|_E \leq C\delta \frac{|y-z|^\delta}{|x-y|^{n+\delta}},$  when  $|x-y| > 2|y-z|$ , for any  $0 < \delta < 2 - \frac{n}{q};$
- (iii) *there exists a constant  $C$  such that for every ball  $B = B(x, s)$  with  $0 < s \leq \frac{1}{2}\rho(x)$ ,*

$$\log\left(\frac{\rho(x)}{s}\right) \frac{1}{|B|} \int_B \|\mathcal{W}_t 1(y) - (\mathcal{W}_t 1)_B\|_E dy \leq C,$$

and, if  $\alpha < \min\{1, 2 - \frac{n}{q}\}$  then

$$\left(\frac{\rho(x)}{s}\right)^\alpha \frac{1}{|B|} \int_B \|\mathcal{W}_t 1(y) - (\mathcal{W}_t 1)_B\|_E dy \leq C.$$

*Proof.* Let us begin with (i). If  $t > |x-y|^2$  then the conclusion is immediate from the estimate of Lemma 4.2. Assume that  $t \leq |x-y|^2$ . Then

$$\begin{aligned} 0 \leq \mathcal{W}_t(x, y) &\leq \frac{C}{|x-y|^n} e^{-c\frac{|x-y|^2}{t}} \left(1 + \frac{\sqrt{t}}{\rho(x)} + \frac{\sqrt{t}}{\rho(y)}\right)^{-N} \\ &= \frac{C}{|x-y|^n} e^{-c\frac{|x-y|^2}{t}} \left(\frac{\sqrt{t}}{|x-y|}\right)^{-N} \left(\frac{|x-y|}{\sqrt{t}} + \frac{|x-y|}{\rho(x)} + \frac{|x-y|}{\rho(y)}\right)^{-N} \\ &\leq \frac{C}{|x-y|^n} e^{-c\frac{|x-y|^2}{t}} \left(\frac{\sqrt{t}}{|x-y|}\right)^{-N} \left(1 + \frac{|x-y|}{\rho(x)} + \frac{|x-y|}{\rho(y)}\right)^{-N} \\ &\leq \frac{C}{|x-y|^n} \left(1 + \frac{|x-y|}{\rho(x)} + \frac{|x-y|}{\rho(y)}\right)^{-N}. \end{aligned}$$

We prove (ii). Observe that if  $|x-y| > 2|y-z|$  then  $|x-y| \sim |x-z|$ . For any  $0 < \delta < \delta_0$ , if  $|y-z| \leq \sqrt{t}$ , by Lemma 4.4,

$$(4.1) \quad |\mathcal{W}_t(x, y) - \mathcal{W}_t(x, z)| \leq C \left(\frac{|y-z|}{\sqrt{t}}\right)^\delta t^{-n/2} e^{-c\frac{|x-y|^2}{t}} \leq C \frac{|y-z|^\delta}{|x-y|^{n+\delta}}.$$

Consider the situation  $|y-z| > \sqrt{t}$ . Then Lemma 4.2 gives

$$|\mathcal{W}_t(x, y)| \leq C \left(\frac{|y-z|}{\sqrt{t}}\right)^\delta t^{-n/2} e^{-c\frac{|x-y|^2}{t}} \left(1 + \frac{\sqrt{t}}{\rho(x)} + \frac{\sqrt{t}}{\rho(y)}\right)^{-N} \leq C \frac{|y-z|^\delta}{|x-y|^{n+\delta}}.$$

The same bound is valid for  $\mathcal{W}_t(x, z)$  because  $|x-z| \sim |x-y|$ . Then the estimate follows directly since  $|\mathcal{W}_t(x, y) - \mathcal{W}_t(x, z)| \leq |\mathcal{W}_t(x, y)| + |\mathcal{W}_t(x, z)|$ . The symmetry of the kernel  $\mathcal{W}_t(x, y) = \mathcal{W}_t(y, x)$  gives the conclusion of (ii).

Let us prove the first statement of (iii). Let  $B = B(x, s)$  with  $0 < s \leq \frac{1}{2}\rho(x)$ . The triangle inequality gives

$$(4.2) \quad \|\mathcal{W}_t 1(y) - (\mathcal{W}_t 1)_B\|_E \leq \frac{1}{|B|} \int_B \|\mathcal{W}_t 1(y) - \mathcal{W}_t 1(z)\|_E dz$$

We estimate the integrand  $\|\mathcal{W}_t 1(y) - \mathcal{W}_t 1(z)\|_E$ . Because  $y, z \in B$ , we have  $\rho(y) \sim \rho(z) \sim \rho(x)$  (see Lemma 2.1). The fact that  $W_t 1(x) \equiv 1$  and Lemma 4.3 entails

$$\begin{aligned}
 |\mathcal{W}_t 1(y) - \mathcal{W}_t 1(z)| &\leq |\mathcal{W}_t 1(y) - W_t 1(y)| + |\mathcal{W}_t 1(z) - W_t 1(z)| \\
 &\leq \int_{\mathbb{R}^n} \left[ \left( \frac{\sqrt{t}}{\rho(y)} \right)^{\delta_0} \omega_t(y-w) + \left( \frac{\sqrt{t}}{\rho(z)} \right)^{\delta_0} \omega_t(z-w) \right] dw \\
 (4.3) \quad &\leq \left( \frac{\sqrt{t}}{\rho(x)} \right)^{\delta_0} \int_{\mathbb{R}^n} [\omega_t(y-w) + \omega_t(z-w)] dw = C \left( \frac{\sqrt{t}}{\rho(x)} \right)^{\delta_0}.
 \end{aligned}$$

So (4.3) gives

$$(4.4) \quad |\mathcal{W}_t 1(y) - \mathcal{W}_t 1(z)| \leq C \left( \frac{s}{\rho(x)} \right)^{\delta_0}, \quad \text{when } \sqrt{t} \leq 2s.$$

If  $\sqrt{t} > 2s$  then  $|y - z| \leq 2s < \sqrt{t}$ . Hence Lemma 4.4 implies that

$$(4.5) \quad |\mathcal{W}_t 1(y) - \mathcal{W}_t 1(z)| \leq \int_{\mathbb{R}^n} |\mathcal{W}_t(y, w) - \mathcal{W}_t(z, w)| dw \leq C \left( \frac{|y - z|}{\sqrt{t}} \right)^{\delta} \leq C \left( \frac{s}{\sqrt{t}} \right)^{\delta},$$

where  $0 < \delta < \delta_0$ . Therefore estimate (4.5) gives

$$(4.6) \quad |\mathcal{W}_t 1(y) - \mathcal{W}_t 1(z)| \leq C \left( \frac{s}{\rho(x)} \right)^{\delta}, \quad \text{when } \sqrt{t} > \rho(x).$$

When  $2s < \sqrt{t} < \rho(x)$  we write

$$\begin{aligned}
 |\mathcal{W}_t 1(y) - \mathcal{W}_t 1(z)| &= |(\mathcal{W}_t 1(y) - W_t 1(y)) - (\mathcal{W}_t 1(z) - W_t 1(z))| \\
 &= \left| \left( \int_{|w-y| > C\rho(y)} + \int_{4|y-z| < |w-y| < C\rho(y)} + \int_{|w-y| < 4|y-z|} \right) \right. \\
 &\quad \left. (\mathcal{W}_t(y, w) - W_t(y, w)) - (\mathcal{W}_t(z, w) - W_t(z, w)) dw \right| \\
 &= |I + II + III|.
 \end{aligned}$$

For  $I$  we use the smoothness proved in part (ii) of this proposition. Note that the same smoothness estimate is valid for the classical heat kernel. So we get

$$|I| \leq C \int_{|w-y| > C\rho(y)} \frac{|y-z|^{\delta}}{|w-y|^{n+\delta}} dw \leq C \left( \frac{s}{\rho(x)} \right)^{\delta}.$$

In  $II$  we apply Lemma 4.5 and the fact that  $\rho(w) \sim \rho(y)$  in the region of integration:

$$|II| \leq C |y-z|^{\delta} \int_{C\rho(y) > |w-y| > 4|y-z|} \frac{\omega_t(w-y)}{\rho(w)^{\delta}} dw \leq C \left( \frac{s}{\rho(x)} \right)^{\delta}.$$

The estimate of  $III$  is obtained by applying Lemma 4.3:

$$\begin{aligned}
 |III| &\leq C \left( \frac{\sqrt{t}}{\rho(x)} \right)^{\delta_0} \left( \int_{|w-y| < 4|y-z|} \omega_t(y-w) dw + \int_{|w-z| \leq 5|y-z|} \omega_t(z-w) dw \right) \\
 &\leq C \left( \frac{\sqrt{t}}{\rho(x)} \right)^{\delta_0} \int_{|\xi| \leq 5 \frac{|y-z|}{\sqrt{t}}} \omega(\xi) d\xi \leq C \left( \frac{\sqrt{t}}{\rho(x)} \right)^{\delta_0} \left( \frac{|y-z|}{\sqrt{t}} \right)^n \leq C \frac{s^n}{\rho(x)^{\delta_0} (\sqrt{t})^{n-\delta_0}} \\
 &\leq C \frac{s^n}{\rho(x)^{\delta_0} s^{n-\delta_0}} = C \left( \frac{s}{\rho(x)} \right)^{\delta_0},
 \end{aligned}$$

since  $2s < \sqrt{t}$  and  $n - \delta_0 > 0$ . Thus

$$(4.7) \quad |\mathcal{W}_t 1(y) - \mathcal{W}_t 1(z)| \leq C \left( \frac{s}{\rho(x)} \right)^{\delta}, \quad \text{when } 2s < \sqrt{t} < \rho(x).$$

Combining (4.4), (4.6) and (4.7), we get

$$(4.8) \quad \|\mathcal{W}_t 1(y) - \mathcal{W}_t 1(z)\|_E \leq C \left( \frac{s}{\rho(x)} \right)^\delta.$$

Therefore, from (4.2) and (4.8) we get

$$\log \left( \frac{\rho(x)}{s} \right) \frac{1}{|B|} \int_B \|\mathcal{W}_t 1(y) - (\mathcal{W}_t 1)_B\|_E \, dy \leq C \left( \frac{s}{\rho(x)} \right)^\delta \log \left( \frac{\rho(x)}{s} \right) \leq C,$$

which is the first conclusion of (iii).

For the second estimate of (iii), by (4.8), we have

$$\left( \frac{\rho(x)}{s} \right)^\alpha \frac{1}{|B|} \int_B \|\mathcal{W}_t 1(y) - (\mathcal{W}_t 1)_B\|_E \, dy \leq C \left( \frac{s}{\rho(x)} \right)^{\delta-\alpha} \leq C,$$

as soon as  $\delta - \alpha \geq 0$ , which can be guaranteed if  $\alpha < \min\{1, 2 - \frac{n}{q}\}$  and we choose  $\delta \geq \alpha$ .  $\square$

**4.2. Maximal operators for the generalized Poisson operators  $\mathcal{P}_t^\sigma$ .** For  $0 < \sigma < 1$  we define the generalized Poisson operators  $\mathcal{P}_t^\sigma$  as

$$(4.9) \quad u(x, t) \equiv \mathcal{P}_t^\sigma f(x) = \frac{t^{2\sigma}}{4^\sigma \Gamma(\sigma)} \int_0^\infty e^{-\frac{t^2}{4r}} \mathcal{W}_r f(x) \frac{dr}{r^{1+\sigma}} = \frac{1}{\Gamma(\sigma)} \int_0^\infty e^{-r} \mathcal{W}_{\frac{t^2}{4r}} f(x) \frac{dr}{r^{1-\sigma}},$$

for  $x \in \mathbb{R}^n$  and  $t > 0$ . The function  $u$  satisfies the following boundary value (extension) problem:

$$\begin{cases} -\mathcal{L}_x u + \frac{1-2\sigma}{t} u_t + u_{tt} = 0, & \text{in } \mathbb{R}^n \times (0, \infty); \\ u(x, 0) = f(x), & \text{on } \mathbb{R}^n. \end{cases}$$

Moreover,  $u$  is useful to characterize the fractional powers of  $\mathcal{L}$  since  $-t^{1-2\sigma} u_t(x, t)|_{t=0} = c_\sigma \mathcal{L}^\sigma f(x)$  for some constant  $c_\sigma > 0$ , see [22]. The fractional powers  $\mathcal{L}^\sigma$  can be defined in a spectral way. When  $\sigma = 1/2$  we get that  $\mathcal{P}_t^{1/2} = e^{-t\mathcal{L}^{1/2}}$  is the classical Poisson semigroup generated by  $\mathcal{L}$  given by Bochner's subordination formula, see [21]. It follows that

$$\mathcal{P}_t^\sigma f(x) = \int_{\mathbb{R}^n} \mathcal{P}_t^\sigma(x, y) f(y) \, dy,$$

where

$$(4.10) \quad \mathcal{P}_t^\sigma(x, y) = \frac{t^{2\sigma}}{4^\sigma \Gamma(\sigma)} \int_0^\infty e^{-\frac{t^2}{4r}} \mathcal{W}_r(x, y) \frac{dr}{r^{1+\sigma}} = \frac{1}{\Gamma(\sigma)} \int_0^\infty e^{-r} \mathcal{W}_{\frac{t^2}{4r}}(x, y) \frac{dr}{r^{1-\sigma}}.$$

To get the boundedness of the maximal operator  $\mathcal{P}^{\sigma,*} f(x) := \sup_{t>0} |\mathcal{P}_t^\sigma f(x)| = \|\mathcal{P}_t^\sigma f(x)\|_E$  in  $BMO_\mathcal{L}^\alpha$ , we proceed using the vector-valued approach and the boundedness of the maximal heat semigroup  $\mathcal{W}^* f$ . The following proposition completely analogous to Proposition 4.6 holds.

**Proposition 4.7.** *The estimates of Proposition 4.6 are valid when  $\mathcal{W}_t$  is replaced by  $\mathcal{P}_t^\sigma$ .*

*Proof.* The proof follows by transferring the estimates for  $\mathcal{W}_t(x, y)$  to  $\mathcal{P}_t^\sigma(x, y)$  through formula (4.10). We just sketch the proof of (iii). For any  $y, z \in B = B(x, s)$ ,  $x \in \mathbb{R}^n$ ,  $0 < s \leq \frac{1}{2}\rho(x)$ , by (4.10), Minkowski's integral inequality and (4.8) we have

$$\begin{aligned} \|\mathcal{P}_t^\sigma 1(y) - \mathcal{P}_t^\sigma 1(z)\|_E &\leq C_\sigma \int_0^\infty t^{2\sigma} e^{-\frac{t^2}{4r}} \|\mathcal{W}_r 1(y) - \mathcal{W}_r 1(z)\|_E \frac{dr}{r^{1+\sigma}} \\ &\leq C \left( \frac{s}{\rho(x)} \right)^\delta \int_0^\infty t^{2\sigma} e^{-\frac{t^2}{4r}} \frac{dr}{r^{1+\sigma}} = C \left( \frac{s}{\rho(x)} \right)^\delta. \end{aligned}$$

Then the same computations for the heat semigroup apply in this case and give (iii).  $\square$

**4.3. Littlewood–Paley  $g$ -function for the heat–diffusion semigroup.** The Littlewood–Paley  $g$ -function associated with  $\{\mathcal{W}_t\}_{t>0}$  is defined by

$$g_{\mathcal{W}}(f)(x) = \left( \int_0^\infty |t\partial_t \mathcal{W}_t f(x)|^2 \frac{dt}{t} \right)^{1/2} = \|t\partial_t \mathcal{W}_t f(x)\|_F,$$

where  $F := L^2((0, \infty), \frac{dt}{t})$ . The Spectral Theorem implies that  $g_{\mathcal{W}}$  is an isometry on  $L^2(\mathbb{R}^n)$ , see [9, Lemma 3]. As before, to get the boundedness of  $g_{\mathcal{W}}$  from  $BMO_{\mathcal{L}}^\alpha$  into itself it is sufficient to prove the following result.

**Proposition 4.8.** *The estimates of Proposition 4.6 are valid when  $\mathcal{W}_t$  is replaced by  $t\partial_t \mathcal{W}_t$  and the Banach space  $E$  is replaced by  $F$ .*

The proof of Proposition 4.8 requires some extra effort. Let us recall the following already well-known estimates.

**Lemma 4.9** (See [9, Proposition 4]). *For any  $N > 0$  there exist constants  $C = C_N$  and  $c > 0$  such that for all  $x, y \in \mathbb{R}^n$ ,  $t > 0$  and  $0 < \delta < \delta_0$ ,*

$$\begin{aligned} (a) \quad & |t\partial_t \mathcal{W}_t(x, y)| \leq Ct^{-n/2} e^{-c\frac{|x-y|^2}{t}} \left( 1 + \frac{\sqrt{t}}{\rho(x)} + \frac{\sqrt{t}}{\rho(y)} \right)^{-N}; \\ (b) \quad & |t\partial_t \mathcal{W}_t(x+h, y) - t\partial_t \mathcal{W}_t(x, y)| \leq C \left( \frac{|h|}{\sqrt{t}} \right)^\delta t^{-n/2} e^{-c\frac{|x-y|^2}{t}} \left( 1 + \frac{\sqrt{t}}{\rho(x)} + \frac{\sqrt{t}}{\rho(y)} \right)^{-N}, \text{ for all } |h| \leq \sqrt{t}; \\ (c) \quad & \left| \int_{\mathbb{R}^n} t\partial_t \mathcal{W}_t(x, y) dy \right| \leq C \frac{(\sqrt{t}/\rho(x))^\delta}{(1 + \sqrt{t}/\rho(x))^N}. \end{aligned}$$

*Proof of Proposition 4.8.* Part (i) is proved using Lemma 4.9(a) and the same argument of the proof of Proposition 4.6(i).

Similarly (ii) follows by Lemma 4.9(b) and the symmetry  $\mathcal{W}_t(x, y) = \mathcal{W}_t(y, x)$ .

To prove (iii) let us fix  $y, z \in B = B(x_0, s)$ ,  $0 < s \leq \frac{1}{2}\rho(x_0)$ . In view of an estimate like (4.2), we must handle  $\|t\partial_t \mathcal{W}_t 1(y) - t\partial_t \mathcal{W}_t 1(z)\|_F$  first. We can write

$$\begin{aligned} (4.11) \quad & \|t\partial_t \mathcal{W}_t 1(y) - t\partial_t \mathcal{W}_t 1(z)\|_F^2 = \int_0^\infty \left| \int_{\mathbb{R}^n} (t\partial_t \mathcal{W}_t(x, y) - t\partial_t \mathcal{W}_t(x, z)) dx \right|^2 \frac{dt}{t} \\ & = \left( \int_0^{4s^2} + \int_{4s^2}^{\rho(x_0)^2} + \int_{\rho(x_0)^2}^\infty \right) \left| \int_{\mathbb{R}^n} (t\partial_t \mathcal{W}_t(x, y) - t\partial_t \mathcal{W}_t(x, z)) dx \right|^2 \frac{dt}{t} =: A_1 + A_2 + A_3. \end{aligned}$$

Since  $y, z \in B \subset B(x_0, \rho(x_0))$ , it follows that  $\rho(y) \sim \rho(x_0) \sim \rho(z)$ . By Lemma 4.9(c),

$$(4.12) \quad A_1 \leq C \int_0^{4s^2} \frac{(\sqrt{t}/\rho(x_0))^{2\delta}}{(1 + \sqrt{t}/\rho(x_0))^{2N}} \frac{dt}{t} \leq C \int_0^{4s^2} \left( \frac{\sqrt{t}}{\rho(x_0)} \right)^{2\delta} \frac{dt}{t} = C \left( \frac{s}{\rho(x_0)} \right)^{2\delta}.$$

Also, by Lemma 4.9(b),

$$\begin{aligned} (4.13) \quad & A_3 \leq C \int_{\rho(x_0)^2}^\infty \left( \frac{|y-z|}{\sqrt{t}} \right)^{2\delta} \left| \int_{\mathbb{R}^n} t^{-n/2} e^{-c\frac{|x-y|^2}{t}} dx \right|^2 \frac{dt}{t} \\ & = C \int_{\rho(x_0)^2}^\infty \left( \frac{|y-z|}{\sqrt{t}} \right)^{2\delta} \frac{dt}{t} \leq C \left( \frac{s}{\rho(x_0)} \right)^{2\delta}. \end{aligned}$$

It remains to estimate the term  $A_2$ . Recall from [9, Eq. (2.8)] that, because the potential  $V$  is in the reverse Hölder class,

$$(4.14) \quad \int_{\mathbb{R}^n} \omega_t(x-y) V(y) dy \leq \frac{C}{t} \left( \frac{\sqrt{t}}{\rho(x)} \right)^\delta, \quad \text{for } t \leq \rho(x)^2.$$

Clearly  $\partial_t \mathcal{W}_t 1(x) = \mathcal{L} \mathcal{W}_t 1(x) = \mathcal{W}_t V(x)$ , that is

$$(4.15) \quad \int_{\mathbb{R}^n} \partial_t \mathcal{W}_t(x, y) dy = \int_{\mathbb{R}^n} \mathcal{W}_t(x, y) V(y) dy.$$

We then have, by Lemma 4.4 (remember that  $|y - z| \leq 2s \leq \sqrt{t}$ ),

$$\begin{aligned}
A_2 &= \int_{4s^2}^{\rho(x_0)^2} \left| \int_{\mathbb{R}^n} (t\partial_t \mathcal{W}_t(x, y) - t\partial_t \mathcal{W}_t(x, z)) \, dx \right|^2 \frac{dt}{t} \\
&= \int_{4s^2}^{\rho(x_0)^2} t \left| \int_{\mathbb{R}^n} (\mathcal{W}_t(y, x) - \mathcal{W}_t(z, x)) V(x) \, dx \right|^2 dt \\
(4.16) \quad &\leq C |y - z|^{2\delta} \int_{4s^2}^{\rho(x_0)^2} t^{1-\delta} \left| \int_{\mathbb{R}^n} t^{-n/2} e^{-c \frac{|y-x|}{t}} V(x) \, dx \right|^2 dt \\
&\leq C s^{2\delta} \int_{4s^2}^{\rho(x_0)^2} t^{1-\delta} t^{-2} \left( \frac{\sqrt{t}}{\rho(y)} \right)^{2\delta} dt \\
&\leq C \left( \frac{s}{\rho(x_0)} \right)^{2\delta} \int_{s^2}^{\rho(x_0)^2} \frac{dt}{t} = C \left( \frac{s}{\rho(x_0)} \right)^{2\delta} \log \left( \frac{\rho(x_0)}{s} \right).
\end{aligned}$$

Combining (4.11), (4.12), (4.13) and (4.16) we get

$$(4.17) \quad \|t\partial_t \mathcal{W}_t 1(y) - t\partial_t \mathcal{W}_t 1(z)\|_F \leq C \left( \frac{s}{\rho(x_0)} \right)^\delta \left( \log \left( \frac{\rho(x_0)}{s} \right) \right)^{1/2}.$$

Thus (iii) readily follows.  $\square$

**4.4. Littlewood–Paley  $g$ -function for the Poisson semigroup.** The Littlewood–Paley  $g$ -function associated with the Poisson semigroup  $\{\mathcal{P}_t\}_{t>0} \equiv \{\mathcal{P}_t^{1/2}\}_{t>0}$  (see (4.9) and (4.10)) is defined analogously as  $g_{\mathcal{W}}$  by replacing the heat semigroup by the Poisson semigroup:

$$g_{\mathcal{P}}(f)(x) = \left( \int_0^\infty |t\partial_t \mathcal{P}_t f(x)|^2 \frac{dt}{t} \right)^{1/2} = \|t\partial_t \mathcal{P}_t f(x)\|_F.$$

The Spectral Theorem shows that  $g_{\mathcal{P}}$  is an isometry on  $L^2(\mathbb{R}^n)$ , see [18, Lemma 3.7]. We also have

**Proposition 4.10.** *The estimates of Proposition 4.6 are valid when  $\mathcal{W}_t$  is replaced by  $t\partial_t \mathcal{P}_t$  and the Banach space  $E$  is replaced by  $F$ .*

*Proof.* First we derive a convenient formula to treat the operator  $t\partial_t \mathcal{P}_t$ . By the second identity of (4.10) with  $\sigma = 1/2$  (Bochner’s subordination formula) and a change of variables,

$$\begin{aligned}
t\partial_t \mathcal{P}_t(x, y) &= \frac{t}{\sqrt{\pi}} \int_0^\infty \frac{e^{-r}}{r^{1/2}} \partial_t \left( \mathcal{W}_{\frac{t^2}{4r}}(x, y) \right) dr = \frac{t^2}{2\sqrt{\pi}} \int_0^\infty \frac{e^{-r}}{r^{1/2}} \partial_v \left( \mathcal{W}_v(x, y) \right) \Big|_{v=\frac{t^2}{4r}} \frac{dr}{r} \\
(4.18) \quad &= \frac{t}{\sqrt{\pi}} \int_0^\infty e^{-\frac{t^2}{4v}} v \partial_v \mathcal{W}_v(x, y) \frac{dv}{v^{3/2}}.
\end{aligned}$$

Formula (4.18) should be compared with the first identity of (4.10) for  $\sigma = 1/2$ . It will allow us to transfer the estimates for  $v\partial_v \mathcal{W}_v$  to  $t\partial_t \mathcal{P}_t$ .

For (i) we use (4.18), Minkowski’s integral inequality and the estimate for  $v\partial_v \mathcal{W}_v$ :

$$\begin{aligned}
\|t\partial_t \mathcal{P}_t(x, y)\|_F^2 &\leq C \int_0^\infty |v\partial_v \mathcal{W}_v(x, y)|^2 \int_0^\infty t e^{-\frac{t^2}{4v}} \frac{dt}{t} \frac{dv}{v^{3/2}} \\
&= C \int_0^\infty |v\partial_v \mathcal{W}_v(x, y)|^2 \frac{dv}{v} \leq \frac{C}{|x - y|^{2n}} \left( 1 + \frac{|x - y|}{\rho(x)} + \frac{|x - y|}{\rho(y)} \right)^{-2N}.
\end{aligned}$$

The estimate for (ii) follows in the same way.

By (4.18), Fubini’s Theorem and (4.17),

$$\|t\partial_t \mathcal{P}_t 1(y) - t\partial_t \mathcal{P}_t 1(z)\|_F \leq C \left( \frac{s}{\rho(x_0)} \right)^\delta \log \left( \frac{\rho(x_0)}{s} \right)^{1/2}.$$

which is sufficient for (iii).  $\square$

**4.5. Laplace transform type multipliers.** Given a bounded function  $a$  on  $[0, \infty)$  we let

$$m(\lambda) = \lambda \int_0^\infty a(t) e^{-t\lambda} dt.$$

The Spectral Theorem allows us to define the Laplace transform type multiplier operator  $m(\mathcal{L})$  associated to  $a$  that is bounded on  $L^2(\mathbb{R}^n)$ . Observe that

$$m(\mathcal{L})f(x) = \int_0^\infty a(t) \mathcal{L} e^{-t\mathcal{L}} f(x) dt = \int_0^\infty a(t) \partial_t \mathcal{W}_t f(x) dt, \quad x \in \mathbb{R}^n.$$

Then the kernel  $\mathcal{M}(x, y)$  of  $m(\mathcal{L})$  can be written as

$$\mathcal{M}(x, y) = \int_0^\infty a(t) \partial_t \mathcal{W}_t(x, y) dt.$$

**Proposition 4.11.** *Let  $x, y, z \in \mathbb{R}^n$ ,  $N > 0$ ,  $0 \leq \alpha < 1$  and  $B = B(x, s)$  for  $0 < s \leq \rho(x)$ . Then*

- (a)  $|\mathcal{M}(x, y)| \leq \frac{C}{|x - y|^n} \left( 1 + \frac{|x - y|}{\rho(x)} + \frac{|x - y|}{\rho(y)} \right)^{-N}$ ;
- (b)  $|\mathcal{M}(x, y) - \mathcal{M}(x, z)| + |\mathcal{M}(y, x) - \mathcal{M}(z, x)| \leq C \delta \frac{|y - z|^\delta}{|x - y|^{n+\delta}}$ , for all  $|x - y| > 2|y - z|$  and any  $0 < \delta < \delta_0$ ;
- (c)  $\log \left( \frac{\rho(x)}{s} \right) \frac{1}{|B|} \int_B |m(\mathcal{L})1(y) - (m(\mathcal{L})1)_B| dy \leq C$ ;
- (d)  $\left( \frac{\rho(x)}{s} \right)^\alpha \frac{1}{|B|} \int_B |m(\mathcal{L})1(y) - (m(\mathcal{L})1)_B| dy \leq C$ , for any  $0 \leq \alpha < \min\{1, 2 - \frac{n}{q}\}$ .

*Proof.* The reader should recall the estimates for  $\partial_t \mathcal{W}_t(x, y)$  stated in Lemma 4.9.

For (a), by Lemma 4.9(a),

$$\begin{aligned} \int_0^{|x-y|^2} |a(t) \partial_t \mathcal{W}_t(x, y)| dt &\leq C \int_0^{|x-y|^2} t^{-n/2} e^{-c \frac{|x-y|^2}{t}} \left( 1 + \frac{\sqrt{t}}{\rho(x)} + \frac{\sqrt{t}}{\rho(y)} \right)^{-N} \frac{dt}{t} \\ &= C \int_0^{|x-y|^2} t^{-n/2} e^{-c \frac{|x-y|^2}{t}} \left( \frac{\sqrt{t}}{|x-y|} \right)^{-N} \left( \frac{|x-y|}{\sqrt{t}} + \frac{|x-y|}{\rho(x)} + \frac{|x-y|}{\rho(y)} \right)^{-N} \frac{dt}{t} \\ &\leq C \int_0^{|x-y|^2} t^{-n/2} e^{-c \frac{|x-y|^2}{t}} \left( 1 + \frac{|x-y|}{\rho(x)} + \frac{|x-y|}{\rho(y)} \right)^{-N} \frac{dt}{t} \\ &\leq \frac{C}{|x-y|^n} \left( 1 + \frac{|x-y|}{\rho(x)} + \frac{|x-y|}{\rho(y)} \right)^{-N}, \end{aligned}$$

and

$$\begin{aligned} \int_{|x-y|^2}^\infty |a(t) \partial_t \mathcal{W}_t(x, y)| dt &\leq C \int_{|x-y|^2}^\infty t^{-n/2} e^{-c \frac{|x-y|^2}{t}} \left( 1 + \frac{\sqrt{t}}{\rho(x)} + \frac{\sqrt{t}}{\rho(y)} \right)^{-N} \frac{dt}{t} \\ &\leq C \int_{|x-y|^2}^\infty t^{-n/2} e^{-c \frac{|x-y|^2}{t}} \left( 1 + \frac{|x-y|}{\rho(x)} + \frac{|x-y|}{\rho(y)} \right)^{-N} \frac{dt}{t} \\ &\leq \frac{C}{|x-y|^n} \left( 1 + \frac{|x-y|}{\rho(x)} + \frac{|x-y|}{\rho(y)} \right)^{-N}. \end{aligned}$$

To check (b) we apply Lemma 4.9(b) to see that

$$\begin{aligned} \int_{|x-y|^2}^\infty |a(t)| |\partial_t \mathcal{W}_t(x, y) - \partial_t \mathcal{W}_t(x, z)| dt &\leq C \int_{|x-y|^2}^\infty \left( \frac{|y-z|}{\sqrt{t}} \right)^\delta t^{-n/2} e^{-c \frac{|x-y|^2}{t}} \frac{dt}{t} \\ &\leq C \frac{|y-z|^\delta}{|x-y|^{n+\delta}}. \end{aligned}$$



Moreover, by Lemma 4.9(a),

$$\int_0^{|x-y|^2} |a(t) \partial_t \mathcal{W}_t(x, y)| \, dt \leq C \int_0^{|x-y|^2} \left( \frac{|y-z|}{\sqrt{t}} \right)^\delta t^{-n/2} e^{-c \frac{|x-y|^2}{t}} \frac{dt}{t} \leq C \frac{|y-z|^\delta}{|x-y|^{n+\delta}}.$$

The same bound is valid for  $\int_0^{|x-y|^2} |a(t)| |\partial_t \mathcal{W}_t(x, z)| \frac{dt}{t}$  because  $|x-z| \sim |x-y|$ . The symmetry of the kernel  $\mathcal{M}(x, y) = \mathcal{M}(y, x)$  gives the conclusion of (b).

Fix  $y, z \in B$ . For (c) and (d), let us estimate the difference

$$|m(\mathcal{L})1(y) - m(\mathcal{L})1(z)| \leq \|a\|_{L^\infty([0, \infty))} \int_0^\infty \left| \int_{\mathbb{R}^n} (\partial_t \mathcal{W}_t(y, w) - \partial_t \mathcal{W}_t(z, w)) \, dw \right| dt.$$

To that end we split the integral in  $t$  into three parts. We start with the part from 0 to  $4s^2$ . From Lemma 4.9(c),

$$\left| \int_0^{4s^2} \int_{\mathbb{R}^n} (\partial_t \mathcal{W}_t(y, w) - \partial_t \mathcal{W}_t(z, w)) \, dw \, dt \right| \leq C \int_0^{4s^2} \left( \frac{\sqrt{t}}{\rho(x)} \right)^\delta \frac{dt}{t} = C \left( \frac{s}{\rho(x)} \right)^\delta.$$

Let us continue with the integral from  $\rho(x)^2$  to  $\infty$ . We apply Lemma 4.9(b):

$$\begin{aligned} \left| \int_{\rho(x)^2}^\infty \int_{\mathbb{R}^n} (\partial_t \mathcal{W}_t(y, w) - \partial_t \mathcal{W}_t(z, w)) \, dw \, dt \right| &\leq C \int_{\rho(x)^2}^\infty \left( \frac{|y-z|}{\sqrt{t}} \right)^\delta \frac{dt}{t} \\ &\leq C \left( \frac{s}{\rho(x)} \right)^\delta. \end{aligned}$$

Finally we consider the part from  $4s^2$  to  $\rho(x)^2$ . Applying (4.15), Lemma 4.4 and (4.14),

$$\begin{aligned} \left| \int_{4s^2}^{\rho(x)^2} \int_{\mathbb{R}^n} (\partial_t \mathcal{W}_t(y, w) - \partial_t \mathcal{W}_t(z, w)) \, dw \, dt \right| &= \int_{4s^2}^{\rho(x)^2} \left| \int_{\mathbb{R}^n} (\mathcal{W}_t(y, w) - \mathcal{W}_t(z, w)) V(w) \, dw \right| dt \\ &\leq C |y-z|^\delta \int_{4s^2}^{\rho(x)^2} \int_{\mathbb{R}^n} t^{-n/2} e^{-c \frac{|y-w|^2}{t}} V(w) \, dw \frac{dt}{t^{\delta/2}} \\ &\leq C \left( \frac{s}{\rho(y)} \right)^\delta \int_{s^2}^{\rho(x)^2} \frac{dt}{t} \leq C \left( \frac{s}{\rho(x)} \right)^\delta \log \left( \frac{\rho(x)}{s} \right). \end{aligned}$$

Hence

$$\begin{aligned} \frac{1}{|B|} \int_B |m(\mathcal{L})1(y) - (m(\mathcal{L})1)_B| \, dy &\leq \frac{C}{s^{2n}} \int_B \int_B |m(\mathcal{L})1(y) - m(\mathcal{L})(z)| \, dy \, dz \\ &\leq C \left( \frac{s}{\rho(x)} \right)^\delta \log \left( \frac{\rho(x)}{s} \right). \end{aligned}$$

Thus (c) is valid and also (d) holds when  $\alpha < \delta$ .  $\square$

**4.6. Riesz transforms.** For every  $i = 1, 2, \dots, n$ , the  $i$ -th Riesz transform  $\mathcal{R}_i$  associated to  $\mathcal{L}$  is defined by

$$\mathcal{R}_i = \partial_{x_i} \mathcal{L}^{-1/2} = \partial_{x_i} \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-t\mathcal{L}} \frac{dt}{t^{1/2}}.$$

We denote by  $\mathcal{R}$  the vector  $\nabla \mathcal{L}^{-1/2} = (\mathcal{R}_1, \dots, \mathcal{R}_n)$ . The Riesz transforms associated to  $\mathcal{L}$  were first studied by Z. Shen in [20]. He showed (Theorem 0.8 of [20]) that if the potential  $V \in RH_q$  with  $q > n$  then  $\mathcal{R}$  is a Calderón–Zygmund operator. In particular, the  $\mathbb{R}^n$ -valued operator  $\mathcal{R}$  is bounded from  $L^2(\mathbb{R}^n)$  into  $L^2_{\mathbb{R}^n}(\mathbb{R}^n)$  and its kernel  $\mathcal{K}$  satisfies, for any  $0 < \delta < 1 - \frac{n}{q}$ ,

$$(4.19) \quad |\mathcal{K}(x, y) - \mathcal{K}(x, z)| + |\mathcal{K}(y, x) - \mathcal{K}(z, x)| \leq C \frac{|y-z|^\delta}{|x-y|^{n+\delta}},$$

whenever  $|x - y| > 2|y - z|$ . Moreover, when  $q > n$  we have that for any  $x, y \in \mathbb{R}^n$ ,  $x \neq y$ , and  $N > 0$  there exists a constant  $C_N$  such that

$$(4.20) \quad |\mathcal{K}(x, y)| \leq \frac{C_N}{|x - y|^n} \left(1 + \frac{|x - y|}{\rho(x)}\right)^{-N},$$

see [20, Eq. (6.5)] and also [5, Lemma 3]. Hence  $\mathcal{R}$  is a  $\gamma$ -Schrödinger-Calderón-Zygmund operator with  $\gamma = 0$ .

The boundedness results of  $\mathcal{R}$  in  $BMO_{\mathcal{L}}^\alpha$  follow by checking the properties of  $\mathcal{R}1$ .

**Proposition 4.12.** *Let  $V \in RH_q$  with  $q > n$  and  $B = B(x_0, s)$  for  $x_0 \in \mathbb{R}^n$  and  $0 < s \leq \frac{1}{2}\rho(x_0)$ . Then*

$$(i) \log \left( \frac{\rho(x_0)}{s} \right) \frac{1}{|B|} \int_B |\mathcal{R}1(y) - (\mathcal{R}1)_B| \, dy \leq C;$$

$$(ii) \left( \frac{\rho(x_0)}{s} \right)^\alpha \frac{1}{|B|} \int_B |\mathcal{R}1(y) - (\mathcal{R}1)_B| \, dy \leq C, \text{ for } \alpha < 1 - \frac{n}{q}.$$

To prove Proposition 4.12, we collect some well-known estimates on  $\mathcal{K}(x, y)$ . Let us denote by  $\mathcal{K}_0$  the kernel of the  $(\mathbb{R}^n$ -valued) classical Riesz transform  $\mathcal{R}_0 = \nabla(-\Delta)^{-1/2}$ .

**Lemma 4.13** ([5, Lemmas 3 and 4]). *Suppose that  $V \in RH_q$  with  $q > n$ .*

(a) *For any  $x, y \in \mathbb{R}^n$ ,  $x \neq y$ ,*

$$|\mathcal{K}(x, y) - \mathcal{K}_0(x, y)| \leq \frac{C}{|x - y|^n} \left( \frac{|x - y|}{\rho(x)} \right)^{2-n/q}.$$

(b) *For any  $0 < \delta < 1 - \frac{n}{q}$  there exists a constant  $C$  such that if  $|z - y| \geq 2|x - y|$  then*

$$|(\mathcal{K}(x, z) - \mathcal{K}_0(x, z)) - (\mathcal{K}(y, z) - \mathcal{K}_0(y, z))| \leq C \frac{|x - y|^\delta}{|z - y|^{n+\delta}} \left( \frac{|z - y|}{\rho(z)} \right)^{2-n/q}.$$

*Proof of Proposition 4.12.* Let  $y, z \in B$ . Then  $\rho(y) \sim \rho(x_0) \sim \rho(z)$ . Since

$$\mathcal{R}1(x) = \lim_{\varepsilon \rightarrow 0^+} \int_{|x-y|>\varepsilon} \mathcal{K}(x, y) \, dy, \quad \text{a.e. } x \in \mathbb{R}^n,$$

we have

$$\begin{aligned} |\mathcal{R}1(y) - \mathcal{R}1(z)| &\leq \lim_{\varepsilon \rightarrow 0^+} \left| \int_{\varepsilon < |x-y| \leq 4\rho(x_0)} \mathcal{K}(y, x) \, dx - \int_{\varepsilon < |x-z| \leq 4\rho(x_0)} \mathcal{K}(z, x) \, dx \right| \\ &\quad + \left| \int_{|x-y|>4\rho(x_0)} \mathcal{K}(y, x) \, dx - \int_{|x-z|>4\rho(x_0)} \mathcal{K}(z, x) \, dx \right| =: \lim_{\varepsilon \rightarrow 0^+} A_\varepsilon + B, \end{aligned}$$

First, let us consider  $A_\varepsilon$ . Since we will consider the limit as  $\varepsilon$  tends to zero, we can assume that  $0 < \varepsilon < 4\rho(x_0) - 2s$ . For every annulus  $E$  we have  $\int_E \mathcal{K}_0(x, y) \, dy = 0$ . Therefore,

$$\begin{aligned} A_\varepsilon &= \left| \int_{\varepsilon < |x-y| \leq 4\rho(x_0)} (\mathcal{K}(y, x) - \mathcal{K}_0(y, x)) \, dx - \int_{\varepsilon < |x-z| \leq 4\rho(x_0)} (\mathcal{K}(z, x) - \mathcal{K}_0(z, x)) \, dx \right| \\ (4.21) \quad &\leq \left| \int_{\mathbb{R}^n} (\mathcal{K}(y, x) - \mathcal{K}_0(y, x)) (\chi_{\varepsilon < |x-y| \leq 4\rho(x_0)}(x) - \chi_{\varepsilon < |x-z| \leq 4\rho(x_0)}(x)) \, dx \right| \\ &\quad + \left| \int_{\mathbb{R}^n} [(\mathcal{K}(y, x) - \mathcal{K}_0(y, x)) - (\mathcal{K}(z, x) - \mathcal{K}_0(z, x))] \chi_{\varepsilon < |x-z| \leq 4\rho(x_0)}(x) \, dx \right| =: A_\varepsilon^1 + A_\varepsilon^2. \end{aligned}$$

The term  $A_\varepsilon^1$  is not zero when  $|\chi_{\varepsilon < |x-y| \leq 4\rho(x_0)}(x) - \chi_{\varepsilon < |x-z| \leq 4\rho(x_0)}(x)| = 1$ , namely, when

- $\varepsilon < |x - y| \leq 4\rho(x_0)$  and  $|x - z| \leq \varepsilon$ ; or
- $\varepsilon < |x - y| \leq 4\rho(x_0)$  and  $|x - z| > 4\rho(x_0)$ ; or
- $\varepsilon < |x - z| \leq 4\rho(x_0)$  and  $|x - y| \leq \varepsilon$ ; or

- $\varepsilon < |x - z| \leq 4\rho(x_0)$  and  $|x - y| > 4\rho(x_0)$ .

In the first case we have  $\varepsilon < |x - y| \leq |x - z| + |z - y| < \varepsilon + 2s$ . Then, by Lemma 4.13(a),

$$(4.22) \quad A_\varepsilon^1 \leq \int_{\varepsilon < |x-y| \leq 2s+\varepsilon} \frac{C}{|x-y|^n} \left( \frac{|x-y|}{\rho(y)} \right)^{2-n/q} dx \leq C \left( \frac{s}{\rho(x_0)} \right)^{2-n/q}.$$

In the second case, by the assumption on  $\varepsilon$ , we get  $\max\{\varepsilon, 4\rho(x_0) - 2s\} = 4\rho(x_0) - 2s < |x - y| \leq 4\rho(x_0)$ . Then Lemma 4.13(a) and the Mean Value Theorem give

$$(4.23) \quad A_\varepsilon^1 \leq \frac{C}{\rho(x_0)^{2-n/q}} \int_{4\rho(x_0)-2s < |x-y| \leq 4\rho(x_0)} |x-y|^{2-n/q-n} dx \leq C \frac{s}{\rho(x_0)}.$$

In the third and fourth cases we obtain the same bounds as in (4.22) and (4.23) by replacing  $y$  by  $z$ . Thus, when  $0 < \delta < 1 - n/q$ ,

$$(4.24) \quad A_\varepsilon^1 \leq C \left( \frac{s}{\rho(x_0)} \right)^\delta.$$

For  $A_\varepsilon^2$ ,

$$(4.25) \quad \begin{aligned} A_\varepsilon^2 &\leq \left| \int_{|x-z| > 2|y-z|} [(\mathcal{K}(y, x) - \mathcal{K}_0(y, x)) - (\mathcal{K}(z, x) - \mathcal{K}_0(z, x))] \chi_{\varepsilon < |x-z| \leq 4\rho(x_0)}(x) dx \right| \\ &\quad + \left| \int_{|x-z| \leq 2|y-z|} [(\mathcal{K}(y, x) - \mathcal{K}_0(y, x)) - (\mathcal{K}(z, x) - \mathcal{K}_0(z, x))] \chi_{\varepsilon < |x-z| \leq 4\rho(x_0)}(x) dx \right| \\ &=: A_\varepsilon^{2,1} + A_\varepsilon^{2,2}. \end{aligned}$$

By Lemma 4.13(b),

$$(4.26) \quad A_\varepsilon^{2,1} \leq C \frac{|y-z|^\delta}{\rho(z)^{2-n/q}} \int_{|x-z| \leq 4\rho(x_0)} |x-z|^{2-n/q-n-\delta} dx \leq C \left( \frac{s}{\rho(x_0)} \right)^\delta.$$

On the other hand, Lemma 4.13(a) gives

$$(4.27) \quad \begin{aligned} A_\varepsilon^{2,2} &\leq \int_{|x-z| \leq 2|y-z|} \frac{C}{|x-y|^n} \left( \frac{|x-y|}{\rho(y)} \right)^{2-n/q} dx + \int_{|x-z| \leq 2|y-z|} \frac{C}{|x-z|^n} \left( \frac{|x-z|}{\rho(z)} \right)^{2-n/q} dx \\ &\leq \frac{C}{\rho(x_0)^{2-n/q}} \left( \int_{|x-y| \leq 3|y-z|} |x-y|^{2-n/q-n} dx + \int_{|x-z| \leq 2|y-z|} |x-z|^{2-n/q-n} dx \right) \\ &\leq C \left( \frac{s}{\rho(x_0)} \right)^{2-n/q} \leq C \left( \frac{s}{\rho(x_0)} \right)^\delta, \end{aligned}$$

for any  $0 < \delta < 1 - n/q$ . Hence, from (4.21), (4.24), (4.25), (4.26) and (4.27) we obtain that for all  $\varepsilon > 0$  sufficiently small,

$$(4.28) \quad A_\varepsilon \leq C \left( \frac{s}{\rho(x_0)} \right)^\delta.$$

Let us now estimate  $B$ . In a similar way,

$$\begin{aligned} B &\leq \int_{|x-y| > 4\rho(x_0)} |\mathcal{K}(y, x) - \mathcal{K}(z, x)| dx \\ &\quad + \int_{\mathbb{R}^n} |\mathcal{K}(z, x)| |\chi_{|x-z| > 4\rho(x_0)}(x) - \chi_{|x-z| > 4\rho(x_0)}(x)| dx =: B_1 + B_2. \end{aligned}$$

In the integrand of  $B_1$  we have  $|x - y| > 4\rho(x_0) \geq 8s > 2|y - z|$ . Therefore the smoothness of the Riesz kernel (4.19) can be applied to get

$$B_1 \leq C \int_{|x-y| > 4\rho(x_0)} \frac{|y-z|^\delta}{|x-y|^{n+\delta}} dx \leq C \left( \frac{s}{\rho(x_0)} \right)^\delta.$$

It is possible to deal with  $B_2$  as with  $A_\varepsilon^1$  above to derive the same bound. Hence,

$$B \leq C \left( \frac{s}{\rho(x_0)} \right)^\delta.$$

This last estimate together with (4.28) imply

$$|\mathcal{R}1(y) - \mathcal{R}1(z)| \leq C \left( \frac{s}{\rho(x_0)} \right)^\delta,$$

where  $0 < \delta < 1 - n/q$ . From here (i) and (ii) readily follow.  $\square$

**4.7. Negative powers.** For any  $\gamma > 0$  the negative powers of  $\mathcal{L}$  are defined as

$$\mathcal{L}^{-\gamma/2} f(x) = \frac{1}{\Gamma(\gamma/2)} \int_0^\infty e^{-t\mathcal{L}} f(x) \frac{dt}{t^{1-\gamma/2}} = \int_{\mathbb{R}^n} \mathcal{K}_\gamma(x, y) f(y) dy,$$

where

$$\mathcal{K}_\gamma(x, y) = \frac{1}{\Gamma(\gamma/2)} \int_0^\infty \mathcal{W}_t(x, y) \frac{dt}{t^{1-\gamma/2}}, \quad x \in \mathbb{R}^n.$$

Therefore, by Lemma 4.2 and a similar argument as in the proof of Proposition 4.6(i), for every  $N > 0$ ,

$$|\mathcal{K}_\gamma(x, y)| \leq \frac{C}{|x - y|^{n-\gamma}} \left( 1 + \frac{|x - y|}{\rho(x)} + \frac{|x - y|}{\rho(y)} \right)^{-N}.$$

In particular,  $\mathcal{L}^{-\gamma/2}$  is bounded from  $L^p(\mathbb{R}^n)$  into  $L^q(\mathbb{R}^n)$ , for  $\frac{1}{q} = \frac{1}{p} - \frac{\gamma}{n}$  with  $1 < p < q < \infty$  and  $0 < \gamma < n$ . Using similar arguments to those in the proof of Proposition 4.6(ii) it can be checked that

$$|\mathcal{K}_\gamma(x, y) - \mathcal{K}_\gamma(x, z)| + |\mathcal{K}_\gamma(y, x) - \mathcal{K}_\gamma(z, x)| \leq C \frac{|y - z|^\delta}{|x - y|^{n-\gamma+\delta}},$$

when  $|x - y| > 2|y - z|$ , for any  $0 < \delta < 2 - \frac{n}{q}$ . Thus  $\mathcal{L}^{-\gamma}$  is a  $\gamma$ -Schrödinger-Calderón-Zygmund operator according to Definition 3.1.

The second item of Theorem 1.4 is a consequence of the following proposition and our two main theorems.

**Proposition 4.14.** *Let  $B = B(x, s)$  with  $0 < s \leq \frac{1}{2}\rho(x)$ . Then*

- (i)  $\log \left( \frac{\rho(x)}{s} \right) \frac{1}{|B|^{1+\frac{\gamma}{n}}} \int_B |\mathcal{L}^{-\gamma/2} 1(y) - (\mathcal{L}^{-\gamma/2} 1)_B| dy \leq C$  if  $\gamma \leq 2 - \frac{n}{q}$ ;
- (ii)  $\left( \frac{\rho(x)}{s} \right)^\alpha \frac{1}{|B|^{1+\frac{\gamma}{n}}} \int_B |\mathcal{L}^{-\gamma/2} 1(y) - (\mathcal{L}^{-\gamma/2} 1)_B| dy \leq C$  if  $\alpha + \gamma < \min\{1, 2 - \frac{n}{q}\}$ .

*Proof.* Fix  $y, z \in B$ , so that  $\rho(x) \sim \rho(y) \sim \rho(z)$ . We can write

$$(4.29) \quad \mathcal{L}^{-\gamma/2} 1(y) - \mathcal{L}^{-\gamma/2} 1(z) = \int_0^\infty \int_{\mathbb{R}^n} (\mathcal{W}_t(y, w) - \mathcal{W}_t(z, w)) dw t^{\gamma/2} \frac{dt}{t}.$$

We split the integral in  $t$  of the difference (4.29) into two parts. From (4.8) we have

$$\left| \int_0^{\rho(x)^2} \int_{\mathbb{R}^n} (\mathcal{W}_t(y, w) - \mathcal{W}_t(z, w)) dw t^{\gamma/2} \frac{dt}{t} \right| \leq C \left( \frac{s}{\rho(x)} \right)^\delta \int_0^{\rho(x)^2} t^{\gamma/2} \frac{dt}{t} = C \left( \frac{s}{\rho(x)} \right)^\delta \rho(x)^\gamma.$$

On the other hand we can use (4.5) to get

$$\left| \int_{\rho(x)^2}^\infty \int_{\mathbb{R}^n} (\mathcal{W}_t(y, w) - \mathcal{W}_t(z, w)) dw t^{\gamma/2} \frac{dt}{t} \right| \leq C \int_{\rho(x)^2}^\infty \left( \frac{s}{\sqrt{t}} \right)^\delta t^{\gamma/2} \frac{dt}{t} \leq C \left( \frac{s}{\rho(x)} \right)^\delta \rho(x)^\gamma,$$

since  $\gamma < \delta$ . An application of these last two estimates to (4.29) finally gives

$$\begin{aligned} \frac{1}{|B|^{1+\frac{n}{q}}} \int_B |\mathcal{L}^{-\gamma/2} 1(y) - (\mathcal{L}^{-\gamma/2} 1)_B| dy &\leq \frac{C}{s^{2n+\gamma}} \int_B \int_B |\mathcal{L}^{-\gamma/2} 1(y) - \mathcal{L}^{-\gamma/2} 1(z)| dy dz \\ &\leq C \left( \frac{s}{\rho(x)} \right)^{\delta-\gamma}. \end{aligned}$$

Thus (i) is valid if  $\gamma < 2 - \frac{n}{q}$  and  $\delta < 2 - \frac{n}{q}$  is chosen such that  $\gamma \leq \delta$ . Also (ii) holds when  $\alpha + \gamma < \min\{1, 2 - \frac{n}{q}\}$ .  $\square$

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